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Lecture #13: Analyticity and the Cauchy-Riemann Equations

Question. Suppose that $f(z) = u(z) + iv(z)$. Under what conditions on $u = u(z) = u(x, y)$ and $v = v(z) = v(x, y)$ is $f(z)$ analytic?

Answer. We certainly need f to be differentiable at z_0. This means that f is defined in some neighbourhood of z_0 and

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \qquad (*)$$

exists. (In particular, the value of the limit is independent of the path $\Delta z \to 0$.) Let $\Delta z = \Delta x + i\Delta y$. We know that $(*)$ exists if (i) $\Delta y = 0$ and $\Delta x \to 0$, and (ii) $\Delta x = 0$ and $\Delta y \to 0$. Consider first the case $\Delta y = 0$. We have

$$\begin{aligned}
\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} \\
&= \frac{f(x_0 + \Delta x + iy_0) - f(x_0 + iy_0)}{\Delta x} \\
&= \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - (u(x_0, y_0) + iv(x_0, y_0))}{\Delta x} \\
&= \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}
\end{aligned}$$

Now consider the case $\Delta x = 0$. We have

$$\begin{aligned}
\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} \\
&= \frac{f(x_0 + iy_0 + i\Delta y) - f(x_0 + iy_0)}{i\Delta y} \\
&= \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - (u(x_0, y_0) + iv(x_0, y_0))}{i\Delta y} \\
&= \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} - i\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}
\end{aligned}$$

Since both of these are expressions for $f'(z_0)$ in the limit, we obtain by equating real and imaginary parts that

$$\lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} = \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y}$$

and

$$\lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} = -\lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}.$$

Equivalently, we find two expressions for $f'(z_0)$, namely

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0)$$

and so

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0).$$

These are the celebrated *Cauchy-Riemann equations.*

Theorem 13.1. *If $f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$ is differentiable at z_0, then the Cauchy-Riemann equations are satisfied at $z_0 = x_0 + iy_0$; that is, if $f'(z_0)$ exists, then*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \textit{and} \quad \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0).$$

This theorem is most useful, however, when considered in the contrapositive.

Corollary 13.2. *Consider $f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$. If the Cauchy-Riemann equations are not satisfied by f at (x_0, y_0), then f is not differentiable at z_0. In particular, if f is not differentiable at z_0, then f is not analytic at z_0.*

Example 13.3. Let $f(z) = \overline{z} = x - iy$ so that

$$u(x, y) = x \quad \text{and} \quad v(x, y) = -y.$$

We find

$$\begin{aligned} \frac{\partial u}{\partial x} &= 1, \quad \frac{\partial v}{\partial y} = -1, \\ \frac{\partial v}{\partial x} &= 0, \quad \frac{\partial u}{\partial y} = 0. \end{aligned}$$

Since the Cauchy-Riemann equations are not satisfied for any z_0, we conclude that f is nowhere differentiable.

Example 13.4. Let $f(z) = |z|^2 = x^2 + y^2$ so that

$$u(x, y) = x^2 \quad \text{and} \quad v(x, y) = y^2.$$

We find

$$\begin{aligned} \frac{\partial u}{\partial x}(x_0, y_0) &= 2x_0, \quad \frac{\partial v}{\partial y}(x_0, y_0) = 0, \\ \frac{\partial v}{\partial x}(x_0, y_0) &= 0, \qquad \frac{\partial u}{\partial y}(x_0, y_0) = 2y_0. \end{aligned}$$

The Cauchy-Riemann equations are only satisfied at $z_0 = (x_0, y_0) = (0, 0)$. Since the Cauchy-Riemann equations are NOT satisfied at $z_0 \neq 0$, we conclude that f is not differentiable at $z_0 \in \mathbb{C} \setminus \{0\}$. Hence, f is not analytic at 0. It is very important to stress that we CANNOT use the Cauchy-Riemann equations to determine whether or not $f'(0)$ exists. (Using the definition of derivative, we showed in Example 12.3 that $f'(0) = 0$.)

Exercise 13.5. Use the Cauchy-Riemann equations to show that $f(z) = \operatorname{Im} z$ is nowhere differentiable.

Exercise 13.6. Use the Cauchy-Riemann equations to show that $f(z) = \operatorname{Re} z$ is nowhere differentiable.

The key observation is that Theorem 13.1 gives us a necessary condition for differentiability, namely if f is differentiable at z_0, then f satisfies the Cauchy-Riemann equations at z_0. It does not, however, give us a sufficient condition for a function to be differentiable. That is, it is possible for a function $f = u + iv$ to satisfy the Cauchy-Riemann equations at z_0, yet not be differentiable at z_0.

Exercise 13.7. Consider the function

$$f(z) = f(x+iy) = \begin{cases} \dfrac{x^{4/3}y^{5/3} + ix^{5/3}y^{4/3}}{x^2+y^2}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Show that the Cauchy-Riemann equations hold at $z = 0$, but that f is not differentiable at $z = 0$. (Hint: Consider $\Delta z \to 0$ along (i) the real axis, and (ii) the line $y = x$.)