Math 312 Fall 2013 Final Exam – Solutions

1. (a) We have $z = \dfrac{2+i}{i-1} = \dfrac{(2+i)(i+1)}{(i-1)(i+1)} = \dfrac{2i+2+i^2+i}{i^2-1} = \dfrac{3i+1}{-2} = -\dfrac{1}{2} - \dfrac{3}{2}i.$

1. (b) Note that $1+i = \sqrt{2}e^{i\pi/4}$ so that $\operatorname{Arg}(1+i) = \pi/4$. This implies $z = \dfrac{1}{2}\log 2 + \dfrac{\pi}{4}i.$

1. (c) We have $z = \sqrt{2}e^{i\pi/3} = \sqrt{2}\,[\cos(\pi/3) + i\sin(\pi/3)] = \dfrac{1}{\sqrt{2}} + \dfrac{\sqrt{3}}{\sqrt{2}}\,i.$

2. (a) We find $u_x(x_0,y_0) = 2(e^{2y_0}+e^{ky_0})\cos(2x_0)$ so that $u_{xx}(x_0,y_0) = -4(e^{2y_0}+e^{ky_0})\sin(2x_0)$ and $u_y(x_0,y_0) = (2e^{2y_0} + ke^{ky_0})\sin(2x_0)$ so that $u_{yy}(x_0,y_0) = (4e^{2y_0} + k^2e^{ky_0})\sin(2x_0)$ which implies $u_{xx}(x_0,y_0) + u_{yy}(x_0,y_0) = 0$ if and only if $k^2 - 4 = 0$. Thus, the required values of k are 2 and -2.

2. (b) If $k \in \{-2, 2\}$ and $f(z) = u(x,y) + iv(x,y)$ is assumed to be analytic, then the Cauchy-Riemann equations imply that $v(x,y)$ satisfies $v_y(x_0,y_0) = 2(e^{2y_0} + e^{ky_0})\cos(2x_0)$ and $v_x(x_0,y_0) = -(2e^{2y_0} + ke^{ky_0})\sin(2x_0)$. From the first equation, we obtain

$$v(x,y) = \left(e^{2y} + \frac{2}{k}e^{ky}\right)\cos(2x) + C_1(x)$$

and from the second equation, we obtain

$$v(x,y) = \frac{1}{2}(2e^{2y} + ke^{ky})\cos(2x) + C_2(y)$$

where C_1 is a function of x only and C_2 is a function of y only. Hence, we obtain the following.

- If $k = -2$, then $v(x,y) = (e^{2y} - e^{-2y})\cos(2x)$, and
- if $k = 2$, then $v(x,y) = 2e^{2y}\cos(2x)$,

3. Observe that if $z = x + iy$, then $\dfrac{f(z) - f(0)}{z - 0} = \dfrac{(\bar{z})^2}{z^2} = \left(\dfrac{x-iy}{x+iy}\right)^2$. We will now show that $\dfrac{f(z)-f(0)}{z-0}$ does not converge as $z \to 0$ by considering two paths approaching 0. First consider $z \to 0$ along the real axis. Thus,

$$\lim_{z\to 0,\, y=0} \left(\frac{x-iy}{x+iy}\right)^2 = \lim_{x\to 0,\, y=0} \left(\frac{x-iy}{x+iy}\right)^2 = \lim_{x\to 0} \left(\frac{x}{x}\right)^2 = 1.$$

Now consider $z \to 0$ along the $y = x$ line. Since

$$\lim_{z\to 0,\, x=y} \left(\frac{x-iy}{x+iy}\right)^2 = \lim_{x\to 0} \left(\frac{x-ix}{x+ix}\right)^2 = \lim_{x\to 0} \left(\frac{1-i}{1+i}\right)^2 = \left(\frac{1-i}{1+i}\right)^2 = -1,$$

we conclude that $f(z)$ is not differentiable at $z = 0$.

(continued)

Remark. If we try to take $z \to 0$ along the imaginary axis, we obtain

$$\lim_{z\to 0,\, x=0} \left(\frac{x-iy}{x+iy}\right)^2 = \lim_{y\to 0,\, x=0} \left(\frac{x-iy}{x+iy}\right)^2 = \lim_{y\to 0} \left(\frac{-iy}{iy}\right)^2 = 1.$$

Thus, this function has the property that the Cauchy-Riemann equations ARE satisfied at 0, but the function is not differentiable at 0.

4. Observe that $f(z) = \dfrac{z-1}{z+1} = \dfrac{z+1-2}{z+1} = 1 - \dfrac{2}{z+1} = h_3 \circ h_2 \circ h_1(z)$ where $h_1(z) = z+1$, $h_2(z) = 1/z$, and $h_3(z) = 1-2z$. If $D = \{z \in \mathbb{C} : |z| < 1\}$ and $D_1 = h_1(D)$, then $D_1 = \{z \in \mathbb{C} : |z-1| < 1\}$. Let $D_2 = h_2(D_1)$. In order to determine D_2, suppose that $z \in D_1$ and $w = 1/z = u + iv$. Hence,

$$|z-1| < 1 \iff |1/w - 1| < 1 \iff |1-w| < |w| \iff (u-1)^2 + v^2 < u^2 + v^2 \iff u > 1/2$$

and so $D_2 = \{z \in \mathbb{C} : \mathrm{Re}(z) > 1/2\}$. Finally, let $D_3 = h_3(D_2) = f(D)$ so that

$$f(D) = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}.$$

5. (a) Since e^z is entire, the Cauchy Integral Formula implies $\displaystyle\int_C \frac{e^z}{z}\,\mathrm{d}z = 2\pi i e^0 = 2\pi i$.

5. (b) If we parametrize C by $z(t) = e^{it}$, $0 \le t \le 2\pi$, then

$$\int_C \frac{e^{|z|}}{z}\,\mathrm{d}z = \int_0^{2\pi} \frac{e^{|e^{it}|}}{e^{it}} \cdot ie^{it}\,\mathrm{d}t = \int_0^{2\pi} ie^1\,\mathrm{d}t = 2\pi e i.$$

5. (c) The Laurent series for $f(z) = z^{-1}e^{1/z}$ valid for $|z| > 0$ is

$$\frac{e^{1/z}}{z} = \sum_{j=0}^{\infty} \frac{z^{-j-1}}{j!} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{2z^3} + \cdots.$$

This implies that

$$\int_C \frac{e^{1/z}}{z}\,\mathrm{d}z = \int_C f(z)\,\mathrm{d}z = 2\pi i\,\mathrm{Res}(f;0) = 2\pi i.$$

5. (d) Since $f(z) = ze^{-z}$ is entire, the Cauchy Integral Theorem implies $\displaystyle\int_C \frac{z}{e^z}\,\mathrm{d}z = 0$.

6. (a) Since

$$f(z) = \frac{\sin(z-i)}{z(z^2+1)(z^2-9)^2} = \frac{\sin(z-i)}{z(z-i)(z+i)(z-3)^2(z+3)^2},$$

we conclude that $z_1 = i$ is a removable singularity, $z_2 = 0$ is a simple pole, $z_3 = -i$ is a simple pole, $z_4 = 3$ is a pole of order two, and $z_5 = -3$ is a pole of order two.

6. (b) Since only z_1, z_2, and z_3 are inside C, we conclude

$$\int_C f(z)\,\mathrm{d}z = 2\pi i\left[\operatorname{Res}(f;z_1)+\operatorname{Res}(f;z_2)+\operatorname{Res}(f;z_3)\right].$$

Since $z_1 = i$ is a removable singularity, $\operatorname{Res}(f;z_1)=0$. Moreover,

$$\operatorname{Res}(f;z_2) = \left.\frac{\sin(z-i)}{(z^2+1)(z^2-9)^2}\right|_{z=0} = \frac{\sin(-i)}{81} = -\frac{\sin(i)}{81}$$

and

$$\operatorname{Res}(f;z_3) = \left.\frac{\sin(z-i)}{z(z-i)(z^2-9)^2}\right|_{z=-i} = \frac{\sin(-2i)}{(-i)(-2i)(i^2-9)^2} = \frac{\sin(2i)}{200}$$

which implies that

$$\int_C f(z)\,\mathrm{d}z = 2\pi i\left[\frac{\sin(2i)}{200} - \frac{\sin(i)}{81}\right].$$

7. Note that

$$f(z) = \frac{1+z}{z^2+z^6} = \frac{1+z}{z^2(1+z^4)} = \frac{1+z}{z^2}\cdot\frac{1}{1+z^4}.$$

If $|z|>1$, then

$$\frac{1}{1+z^4} = \frac{1/z^4}{1+1/z^4} = \frac{1}{z^4}\sum_{j=0}^{\infty}(-1)^j z^{-4j} = \sum_{j=0}^{\infty}(-1)^j z^{-4-4j}$$

so that

$$\begin{aligned} f(z) &= \frac{1+z}{z^2}\sum_{j=0}^{\infty}(-1)^j z^{-4-4j} = \sum_{j=0}^{\infty}(-1)^j z^{-6-4j} + \sum_{j=0}^{\infty}(-1)^j z^{-5-4j} \\ &= \left[z^{-6} - z^{-10} + z^{-14} - z^{-18} + \cdots\right] + \left[z^{-5} - z^{-9} + z^{-13} - z^{-17} + \cdots\right] \\ &= z^{-5} + z^{-6} - z^{-9} - z^{-10} + z^{-13} + z^{-14} - z^{-17} - z^{-18} + \cdots. \end{aligned}$$

8. If $C = \{|z|=1\}$ denotes the unit circle parametrized by $z(\theta)=e^{i\theta}$, $0\le\theta\le 2\pi$, then

$$\int_0^{2\pi}\frac{1}{1+\sin^2\theta}\,\mathrm{d}\theta = \int_C \frac{1}{1+(z-1/z)^2/(2i)^2}\cdot\frac{1}{iz}\,\mathrm{d}z = 4i\int_C\frac{z}{(z^2-1)^2-4z^2}\,\mathrm{d}z.$$

Note that $(z^2-1)^2-4z^2$ is the difference of perfect squares so that

$$(z^2-1)^2-4z^2 = (z^2-1-2z)(z^2-1+2z).$$

We now write $z^2-2z-1=(z-z_1)(z-z_2)$ where $z_1 = 1+\sqrt{2}$ and $z_2 = 1-\sqrt{2}$, as well as $z^2+2z-1=(z-z_3)(z-z_4)$ where $z_3=-1+\sqrt{2}$ and $z_4 = -1-\sqrt{2}$, and note that only

z_2 and z_3 are inside C. By the Cauchy Residue Theorem,

$$\begin{aligned}
\int_C \frac{z}{(z^2-1)^2-4z^2}\,\mathrm{d}z &= \int_C \frac{z}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}\,\mathrm{d}z \\
&= 2\pi i\left[\frac{z_2}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} + \frac{z_3}{(z_3-z_1)(z_3-z_2)(z_3-z_4)}\right] \\
&= 2\pi i\left[\frac{1-\sqrt{2}}{(-2\sqrt{2})(2-2\sqrt{2})(2)} + \frac{-1+\sqrt{2}}{(-2)(-2+2\sqrt{2})(2\sqrt{2})}\right] \\
&= 2\pi i\left[-\frac{1}{8\sqrt{2}} - \frac{1}{8\sqrt{2}}\right] \\
&= -\frac{\pi i}{2\sqrt{2}}
\end{aligned}$$

and so

$$\int_0^{2\pi} \frac{1}{1+\sin^2\theta}\,\mathrm{d}\theta = 4i\int_C \frac{z}{(z^2-1)^2-4z^2}\,\mathrm{d}z = 4i\cdot -\frac{\pi i}{2\sqrt{2}} = \sqrt{2}\,\pi.$$

9. The basic error with the reasoning in the problem has to do with the definition of square root of a complex variable. If x is a non-negative real number, then $x^{1/2}$ is defined to be the **unique** non-negative real number y such that $y^2 = x$. In other words, we define $x^{1/2} = \sqrt{x}$. If z is any complex variable which is not purely real with non-negative real part, then $z^{1/2}$ describes a set, namely the set of **all** complex variables w such that $w^2 = z$. In fact, there are always two distinct such values. Thus, the error in the problem is that $(-1)^{1/2}$ is being used, on the one hand to represent one of its values, and on the other hand to represent its other value; that is, the problem incorrectly writes $e^{i\pi/2} = (-1)^{1/2} = e^{-i\pi/2}$ and deduces the contradiction instead of writing $(-1)^{1/2} = \{e^{i\pi/2}, e^{-i\pi/2}\}$.

10. In order to prove that $f(z)$ has an isolated singular point at 0, note that if $z \neq 0$, then $f(z) = \dfrac{e^{1/z}\sin z}{z^2}$ is the ratio of an analytic function to a non-zero analytic function and is therefore analytic. Hence, $f(z)$ is analytic everywhere except 0 implying that $f(z)$ has an isolated singular point at 0.

In order to classify the isolated singular point at 0, recall from class that a function $f(z)$ has a pole of order 2 at 0 if and only if

$$f(z) = \frac{g(z)}{z^2}$$

for some analytic function $g(z)$ satisfying $g(0) \neq 0$. Since

$$f(z) = \frac{e^{1/z}\sin z}{z^2}$$

and since $g(z) = e^{1/z}\sin z$ is not analytic at 0, we conclude immediately that $f(z)$ does NOT have a pole at 0. This means that 0 is either a removable singlularity or an essential singularity. In order to determine which type it is, we must consider the Laurent series for

$f(z)$. If $|z| > 0$, then

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{(2j+1)!}$$

and

$$\frac{1}{z} e^{1/z} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{2! z^3} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k! z^{k+1}}$$

so that

$$f(z) = \left[\sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{(2j+1)!} \right] \left[\sum_{k=0}^{\infty} \frac{1}{k! z^{k+1}} \right].$$

Suppose now that the Laurent series for $f(z)$ is given by

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n.$$

By definition, $f(z)$ has a removable singularity at 0 if $c_n = 0$ for all $n < 0$. If $c_n \neq 0$ for only finitely many $n < 0$, then $f(z)$ has a pole at 0, whereas if $c_n \neq 0$ for infinitely many $n < 0$, then $f(z)$ has an essential singularity at 0. Thus, since we already know that $f(z)$ does NOT have a pole at 0, if we can show $c_n \neq 0$ for at least one $n < 0$, then we can conclude that $f(z)$ has an essential singularity at 0. We will show that $c_{-1} \neq 0$. Basically, one needs to multiply the two series together and keep track of which products give a contribution to the z^{-1} term. That is,

$$c_{-1} z^{-1} = 1 \cdot \frac{1}{z} - \frac{z^2}{3!} \frac{1}{2! z^3} + \frac{z^4}{5!} \frac{1}{4! z^5} - \cdots = z^{-1} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!(2j+1)!}.$$

Since $c_{-1} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!(2j+1)!} \neq 0$, we conclude that 0 is an essential singularity. Note that the expression for c_{-1} is a special case of a *Kelvin function*, named after Lord Kelvin (of absolute zero temperature fame), and occurs in the study of cylindrical harmonics.