University of Regina
Department of Mathematics & Statistics
Final Examination
201330

Mathematics 312
Complex Analysis I

Instructor: Michael Kozdron

This exam has 10 problems and 2 numbered pages.

This exam is worth 150 points. The number of points per problem is indicated. For problems with multiple parts, all parts are equally weighted.

You have 3 hours to complete this exam. Please read all instructions carefully, and check your answers. Write your solutions in the exam booklets. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements.*

You are permitted to use one $8\frac{1}{2} \times 11$ *double sided page of notes. No other aids are allowed.*

You must hand in all exam booklets containing solutions, although you may keep the problems.

1. (*12 points*) For each of the following complex variables z, determine values of real numbers a and b such that $z = a + ib$.

(a) $z = \dfrac{2+i}{i-1}$

(b) $z = \mathrm{Log}(1+i)$

(c) $z = \sqrt{2}e^{i\pi/3}$

2. (*16 points*) Let $z = x + iy$ be a complex variable.

(a) Determine all values of $k \in \mathbb{Z}$ (i.e., all integer values of k) such that

$$u(z) = u(x, y) = (e^{2y} + e^{ky})\sin(2x)$$

is harmonic in the plane.

(b) For each value of k that you found in **(a)**, determine a function $v(z) = v(x, y)$ such that

$$f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$$

is analytic in the plane.

3. (*12 points*) Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \begin{cases} \dfrac{(\overline{z})^2}{z}, & z \neq 0, \\ 0, & z = 0. \end{cases}$

Prove that $f(z)$ is not differentiable at $z = 0$.

4. (*12 points*) Let $D = \{z \in \mathbb{C} : |z| < 1\}$ denote the open disk of radius 1 centred at the origin in the complex plane, and consider the function $f : D \to \mathbb{C}$ given by

$$w = f(z) = \frac{z-1}{z+1}.$$

Determine $f(D)$, the image of D under f. Express your answer both (a) analytically, and (b) graphically. Note that a simple sketch highlighting the key features of $f(D)$ will suffice for (b).

5. (*24 points*) Suppose that $C = \{|z| = 1\}$ denotes the circle of radius 1 centred at 0 oriented counterclockwise. Compute the following contour integrals. You may use whichever method you find most appropriate for each integral.

(a) $\displaystyle\int_C \frac{e^z}{z}\,\mathrm{d}z,$

(b) $\displaystyle\int_C \frac{e^{|z|}}{z}\,\mathrm{d}z,$

(c) $\displaystyle\int_C \frac{e^{1/z}}{z}\,\mathrm{d}z,$

(d) $\displaystyle\int_C \frac{z}{e^z}\,\mathrm{d}z.$

6. (*16 points*) Suppose that

$$f(z) = \frac{\sin(z-i)}{z(z^2+1)(z^2-9)^2}.$$

(a) Find all isolated singular points of $f(z)$, and classify them into the three types (removable, pole, or essential). For the poles, determine the orders.

(b) Compute

$$\int_C f(z)\,\mathrm{d}z$$

where $C = \{|z| = 2\}$ denotes the circle of radius 2 centred at 0 oriented counterclockwise.

7. (*16 points*) Determine the Laurent series of

$$f(z) = \frac{1+z}{z^2+z^6}$$

in powers of z that converges for $|z| > 1$.

8. (*16 points*) Compute

$$\int_0^{\pi/4} \frac{1}{1+\sin^2\theta}\,\mathrm{d}\theta.$$

9. (*10 points*) What is wrong with the following reasoning? Explain.

$$i = (-1)^{1/2} = \left((-1)^3\right)^{1/2} = (-1)^{3/2} = \left((-1)^{1/2}\right)^3 = i^3 = -i \quad \text{so} \quad 1 = -1 \quad \text{implying} \quad 1 = 0.$$

10. (*16 points*) Consider the functions $f_1(z)$, $f_2(z)$, and $f_3(z)$ defined by

$$f_1(z) = \frac{1}{z}, \quad f_2(z) = \frac{\sin z}{z}, \quad f_3(z) = \frac{1}{e^{1/z}},$$

respectively. Note that 0 is an isolated singular point for each of these functions. In fact, 0 is a simple pole for $f_1(z)$, a removable singularity for $f_2(z)$, and an essential singularity for $f_3(z)$. Now consider the function $f(z) = f_1(z)f_2(z)f_3(z)$ so that

$$f(z) = \frac{\sin z}{z^2 e^{1/z}}.$$

Prove that 0 is an isolated singular point for $f(z)$ and determine its type (either removable, pole, or essential).