University of Regina
Department of Mathematics & Statistics
Final Examination
201230

Mathematics 312
Complex Analysis I

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This exam has 8 problems and 2 numbered pages.

This exam is worth 150 points. The number of points per problem is indicated. For problems with multiple parts, all parts are equally weighted.

You have 50 minutes to complete this exam. Please read all instructions carefully, and check your answers. Write your solutions in the exam booklets. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements.*

You are permitted to use one $8\frac{1}{2} \times 11$ *double sided page of notes. No other aids are allowed.*

You must hand in all exam booklets containing solutions, although you may keep the problems.

1. (*16 points*) Find all complex variables $z \in \mathbb{C}$ with $|z| < 10$ that satisfy $e^z + 2e^{-z} = 3$.

2. (*20 points*) Consider the mapping $w = e^{-iz}$ of the domain

$$D = \left\{ z \in \mathbb{C} : \mathrm{Im}(z) < 0,\ -\frac{\pi}{2} < \mathrm{Re}(z) < 0 \right\}.$$

Determine the image of D in the w-plane.

3. (*24 points*) Calculate the Laurent series for

$$f(z) = \frac{z}{z^2 - 4z + 3}$$

in powers of z that converges in the domains

(i) $|z| < 1$,

(ii) $1 < |z| < 3$, and

(iii) $|z| > 3$.

4. (*16 points*) Compute

$$\int_C \frac{1}{(z^2+1)^3}\, \mathrm{d}z$$

where $C = \{|z - i| = 1\}$ is the circle of radius 1 centred at i oriented counterclockwise.

5. (*16 points*) Compute

$$\int_0^{2\pi} \frac{\cos\theta}{5 - 4\cos\theta}\, \mathrm{d}\theta.$$

6. (*18 points*) **(a)** Recall from the Fundamental Theorem of Algebra that any polynomial of degree n in a complex variable z has exactly n roots (counting multiplicity). Using the fact that $(z-1)(z^4+z^3+z^2+z+1) = z^5-1$, determine the four roots of the polynomial

$$P(z) = z^4 + z^3 + z^2 + z + 1.$$

(b) Suppose that

$$f(z) = \frac{z^2 - z}{z^9 - z^4}.$$

Find all isolated singular points of $f(z)$, and classify them into the three types (removable, pole, or essential). For the poles, determine the orders.

7. (*16 points*) Compute

$$\int_C z^3 e^{-1/z}\, \mathrm{d}z$$

where $C = \{|z| = 1/2\}$ is oriented counterclockwise.

8. (*24 points*) Consider the function $g : \mathbb{C} \to \mathbb{C}$ given by

$$g(w) = \begin{cases} \dfrac{\sin w}{w}, & w \neq 0, \\ w_0, & w = 0. \end{cases}$$

where $w_0 \in \mathbb{C}$ is a constant.

(a) Determine the unique value of w_0 for which $g(w)$ is analytic.

(b) Prove that $g(w) \neq 0$ for all $|w| \leq 3$.

(c) Suppose that C denotes the unit circle centred at the origin oriented counterclockwise. Suppose further that $f : \mathbb{C} \to \mathbb{C}$ is entire. Prove that if $|z| < 1$, then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\sin(\zeta - z)}\, \mathrm{d}\zeta.$$