Mathematics 312 (Fall 2012) October 26, 2012
Prof. Michael Kozdron

Lecture #22: The Cauchy Integral Formula

Recall that the Cauchy Integral Theorem, Basic Version states that if D is a domain and $f(z)$ is analytic in D with $f'(z)$ continuous, then

$$\int_C f(z)\,\mathrm{d}z = 0$$

for any closed contour C lying entirely in D having the property that C is continuously deformable to a point.

We also showed that if C is any closed contour oriented counterclockwise in $\mathbb{C}$ and a is inside C, then

$$\int_C \frac{1}{z-a}\,\mathrm{d}z = 2\pi i. \tag{$*$}$$

Our goal now is to derive the celebrated Cauchy Integral Formula which can be viewed as a generalization of $(*)$.

Theorem 22.1 (Cauchy Integral Formula)**.** *Suppose that D is a domain and that $f(z)$ is analytic in D with $f'(z)$ continuous. If C is a closed contour oriented counterclockwise lying entirely in D having the property that the region surrounded by C is a simply connected subdomain of D (i.e., if C is continuously deformable to a point) and a is inside C, then*

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,\mathrm{d}z.$$

Proof. Observe that we can write

$$\int_C \frac{f(z)}{z-a}\,\mathrm{d}z = \int_C \frac{f(a)}{z-a}\,\mathrm{d}z + \int_C \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z = 2\pi f(a)i + \int_{C_a} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z$$

where $C_a = \{|z-a| = r\}$ oriented counterclockwise since $(*)$ implies

$$\int_C \frac{f(a)}{z-a}\,\mathrm{d}z = f(a)\int_C \frac{1}{z-a}\,\mathrm{d}z = 2\pi f(a)i$$

and

$$\int_C \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z = \int_{C_a} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z$$

since the integrand

$$\frac{f(z)-f(a)}{z-a}$$

is analytic everywhere except at $z=a$ and its derivative is continuous everywhere except at $z=a$ so that integration over C can be continuously deformed to integration over C_a. However, if we write

$$\int_C \frac{f(z)}{z-a}\,\mathrm{d}z - 2\pi f(a)i = \int_{C_a} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z,$$

and note that the left side of the previous expression does not depend on r, then we conclude

$$\int_C \frac{f(z)}{z-a}\,\mathrm{d}z - 2\pi f(a)i = \lim_{r\downarrow 0}\int_{C_a}\frac{f(z)-f(a)}{z-a}\,\mathrm{d}z.$$

Hence, the proof will be complete if we can show that

$$\lim_{r\downarrow 0}\int_{C_a}\frac{f(z)-f(a)}{z-a}\,\mathrm{d}z = 0.$$

To this end, suppose that $M_r = \max\{|f(z)-f(a)|,\ z \text{ on } C_a\}$. Therefore, if z is on $C_a = \{|z-a| = r\}$, then

$$\left|\frac{f(z)-f(a)}{z-a}\right| = \frac{|f(z)-f(a)|}{|z-a|} = \frac{|f(z)-f(a)|}{r} \le \frac{M_r}{r}$$

so that

$$\begin{aligned}\left|\int_{C_a}\frac{f(z)-f(a)}{z-a}\,\mathrm{d}z\right| \le \int_{C_a}\left|\frac{f(z)-f(a)}{z-a}\right|\,\mathrm{d}z \le \int_{C_a}\frac{M_r}{r}\,\mathrm{d}z = \frac{M_r}{r}\int_{C_a}1\,\mathrm{d}z &= \frac{M_r}{r}\ell(C_a)\\ &= \frac{M_r}{r}\cdot 2\pi r\\ &= 2\pi M_r\end{aligned}$$

since the arclength of C_a is $\ell(C_a) = 2\pi r$. However, since $f(z)$ is analytic in D, we know that $f(z)$ is necessarily continuous in D so that

$$\lim_{z\to a}|f(z)-f(a)| = 0 \quad \text{or, equivalently,} \quad \lim_{r\downarrow 0} M_r = 0.$$

Therefore,

$$\lim_{r\downarrow 0}\left|\int_{C_a}\frac{f(z)-f(a)}{z-a}\,\mathrm{d}z\right| \le \lim_{r\downarrow 0}(2\pi M_r) = 0$$

as required. □

Example 22.2. Compute

$$\frac{1}{2\pi i}\int_C \frac{ze^z}{z-i}\,\mathrm{d}z$$

where $C = \{|z| = 2\}$ is the circle of radius 2 centred at 0 oriented counterclockwise.

Solution. Observe that $f(z) = ze^z$ is entire, $f'(z) = ze^z + e^z$ is continuous, and i is inside C. Therefore, by the Cauchy Integral Formula,

$$\frac{1}{2\pi i}\int_C \frac{ze^z}{z-i}\,\mathrm{d}z = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-i}\,\mathrm{d}z = f(i) = ie^i.$$

Example 22.3. Compute

$$\int_C \frac{ze^z}{z+i}\,\mathrm{d}z$$

where $C = \{|z| = 2\}$ is the circle of radius 2 centred at 0 oriented counterclockwise.

Solution. Observe that $f(z) = ze^z$ is entire, $f'(z) = ze^z + e^z$ is continuous, and $-i$ is inside C. Therefore, by the Cauchy Integral Formula,

$$\int_C \frac{ze^z}{z+i}\,\mathrm{d}z = \int_C \frac{f(z)}{z+i}\,\mathrm{d}z = 2\pi i f(-i) = 2\pi i \cdot -ie^{-i} = 2\pi e^{-i}.$$

Example 22.4. Compute

$$\int_C \frac{ze^z}{z^2+1}\,\mathrm{d}z$$

where $C = \{|z| = 2\}$ is the circle of radius 2 centred at 0 oriented counterclockwise.

Solution. Observe that partial fractions implies

$$\frac{1}{z^2+1} = \frac{1}{z^2-i^2} = \frac{1}{(z+i)(z-i)} = \frac{i/2}{z+i} - \frac{i/2}{z-i}$$

and so

$$\int_C \frac{ze^z}{z^2+1}\,\mathrm{d}z = \frac{i}{2}\int_C \frac{ze^z}{z+i}\,\mathrm{d}z - \frac{i}{2}\int_C \frac{ze^z}{z-i}\,\mathrm{d}z.$$

Let $f(z) = ze^z$. Note that $f(z)$ is entire and $f'(z) = ze^z + e^z$ is continuous. Since both i and $-i$ are inside C, the Cauchy Integral Formula implies

$$\int_C \frac{ze^z}{z+i}\,\mathrm{d}z = 2\pi i f(-i) = 2\pi i \cdot -ie^{-i} = 2\pi e^{-i} \text{ and } \int_C \frac{ze^z}{z-i}\,\mathrm{d}z = 2\pi i f(i) = 2\pi i \cdot ie^{i} = -2\pi e^{i}$$

so that

$$\int_C \frac{ze^z}{z^2+1}\,\mathrm{d}z = \frac{i}{2}\cdot 2\pi e^{-i} - \frac{i}{2}\cdot -2\pi e^{i} = \pi i e^{-i} + \pi i e^{i} = 2\pi i\left[\frac{e^i + e^{-i}}{2}\right] = 2\pi i \cos 1.$$

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Lecture #23: Consequences of the Cauchy Integral Formula

The main result that we will establish today is that an analytic function has derivatives of all orders. The key to establishing this is to first prove a slightly more general result.

Theorem 23.1. *Let g be continuous on the contour C and for each z_0 not on C, set*

$$G(z_0) = \int_C \frac{g(\zeta)}{\zeta - z_0}\,\mathrm{d}\zeta.$$

Then G is analytic at z_0 with

$$G'(z_0) = \int_C \frac{g(\zeta)}{(\zeta - z_0)^2}\,\mathrm{d}\zeta. \tag{$*$}$$

Remark. Observe that in the statement of the theorem, we do not need to assume that g is analytic or that C is a closed contour.

Proof. Let z_0 not on C be fixed. In order to prove the differentiability of G and the desired formula for $G'(z_0)$, we must show that

$$\lim_{\Delta z \to 0} \frac{G(z_0 + \Delta z) - G(z_0)}{\Delta z} = \int_C \frac{g(\zeta)}{(\zeta - z_0)^2}\,\mathrm{d}\zeta.$$

Observe that

$$\begin{aligned}
\frac{G(z_0 + \Delta z) - G(z_0)}{\Delta z} &= \frac{1}{\Delta z}\int_C \frac{g(\zeta)}{\zeta - (z_0 + \Delta z)} - \frac{g(\zeta)}{\zeta - z_0}\,\mathrm{d}\zeta \\
&= \frac{1}{\Delta z}\int_C g(\zeta)\left[\frac{1}{\zeta - (z_0 + \Delta z)} - \frac{1}{\zeta - z_0}\right]\mathrm{d}\zeta \\
&= \int_C \frac{g(\zeta)}{(\zeta - z_0 - \Delta z)(\zeta - z_0)}\,\mathrm{d}\zeta
\end{aligned}$$

and so (with a bit of algebra)

$$\begin{aligned}
\frac{G(z_0 + \Delta z) - G(z_0)}{\Delta z} - \int_C \frac{g(\zeta)}{(\zeta - z_0)^2}\,\mathrm{d}\zeta &= \int_C \frac{g(\zeta)}{(\zeta - z_0 - \Delta z)(\zeta - z_0)}\,\mathrm{d}\zeta - \int_C \frac{g(\zeta)}{(\zeta - z_0)^2}\,\mathrm{d}\zeta \\
&= \Delta z \int_C \frac{g(\zeta)}{(\zeta - z_0 - \Delta z)(\zeta - z_0)^2}\,\mathrm{d}\zeta
\end{aligned} \tag{$\dagger$}$$

The next step is to show that

$$\left|\int_C \frac{g(\zeta)}{(\zeta - z_0 - \Delta z)(\zeta - z_0)^2}\,\mathrm{d}\zeta\right|$$

is bounded.

To this end, let $M = \max\{|g(\zeta)| : \zeta \in C\}$ be the maximum value of $|g(\zeta)|$ on C, and let $d = \min\{\operatorname{dist}(z_0, w) : w \in C\}$ be the minimal distance from z_0 to C. Note that $|\zeta - z_0| \geq d > 0$ for all ζ on C. Without loss of generality, assume that $|\Delta z| < d/2$ (since we ultimately care about $\Delta z \to 0$, this is a valid assumption). By the triangle inequality, if $\zeta \in C$, then

$$|\zeta - z_0 - \Delta z| \geq |\zeta - z_0| - |\Delta z| \geq d - \frac{d}{2} = \frac{d}{2},$$

and so

$$\begin{aligned}\left|\int_C \frac{g(\zeta)}{(\zeta - z_0 - \Delta z)(\zeta - z_0)^2}\,\mathrm{d}\zeta\right| \leq \int_C \left|\frac{g(\zeta)}{(\zeta - z_0 - \Delta z)(\zeta - z_0)^2}\right|\,\mathrm{d}\zeta &\leq \frac{M}{\frac{d}{2}\cdot d^2}\int_C 1\,\mathrm{d}\zeta \\ &= \frac{2M\ell(C)}{d^3}\end{aligned}$$

where $\ell(C) < \infty$ is the arclength of the contour C. Hence, considering (†), we find

$$\begin{aligned}\lim_{\Delta z \to 0}\left|\frac{G(z_0 + \Delta z) - G(z_0)}{\Delta z} - \int_C \frac{g(\zeta)}{(\zeta - z_0)^2}\,\mathrm{d}\zeta\right| &= \lim_{\Delta z \to 0}\left|\Delta z \int_C \frac{g(\zeta)}{(\zeta - z_0 - \Delta z)(\zeta - z_0)^2}\,\mathrm{d}\zeta\right| \\ &\leq \frac{2M\ell(C)}{d^3}\lim_{\Delta z \to 0}|\Delta z| \\ &= 0\end{aligned}$$

so that $(*)$ holds. Note that we have proved $G(z_0)$ is differentiable at $z_0 \notin C$ for z_0 fixed. Since z_0 was arbitrary, we conclude that $G(z_0)$ is differentiable at *any* $z_0 \notin C$ implying that G is analytic at $z_0 \notin C$ as required. □

It is important to note that exactly the same method of proof yields the following result.

Corollary 23.2. *Let g be continuous on the contour C and for each z_0 not on C, set*

$$H(z_0) = \int_C \frac{g(\zeta)}{(\zeta - z_0)^n}\,\mathrm{d}\zeta$$

where n is a positive integer. Then H is analytic at z_0 with

$$H'(z_0) = n\int_C \frac{g(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathrm{d}\zeta. \qquad (**)$$

Now we make a very important observation that follows immediately from Theorem 23.1 and Corollary 23.2.

Theorem 23.3. *If $f(z)$ is analytic in a domain D, then all of its derivatives $f'(z)$, $f''(z)$, $f'''(z)$, ... exist and are themselves analytic.*

Remark. This theorem is remarkable because it is unique to complex analysis. The analogue for real-valued functions is not true. For example, $f(x) = 9x^{5/3}$ for $x \in \mathbb{R}$ is differentiable for all x, but its derivative $f'(x) = 15x^{2/3}$ is not differentiable at $x = 0$ (i.e., $f''(x) = 10x^{-1/3}$ does not exist when $x = 0$).

Moreover, if the function in the statement of Theorem 23.1 happens to be analytic and C happens to be a closed contour oriented counterclockwise, then we arrive at the following important theorem which might be called the General Version of the Cauchy Integral Formula.

Theorem 23.4 (Cauchy Integral Formula, General Version)**.** *Suppose that $f(z)$ is analytic inside and on a simply closed contour C oriented counterclockwise. If z is any point inside C, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta,$$

$n = 1, 2, 3, \ldots$.

For the purposes of computations, it is usually more convenient to write the General Version of the Cauchy Integral Formula as follows.

Corollary 23.5. *Suppose that $f(z)$ is analytic inside and on a simply closed contour C oriented counterclockwise. If a is any point inside C, then*

$$\int_C \frac{f(z)}{(z-a)^m}\,\mathrm{d}z = \frac{2\pi i f^{(m-1)}(a)}{(m-1)!}.$$

Example 23.6. Compute

$$\int_C \frac{e^{5z}}{(z-i)^3}\,\mathrm{d}z$$

where $C = \{|z| = 2\}$ is the circle of radius 2 centred at 0 oriented counterclockwise.

Solution. Let $f(z) = e^{5z}$ so that $f(z)$ is entire, and let $a = i$ which is inside C. Therefore,

$$\int_C \frac{e^{5z}}{(z-i)^3}\,\mathrm{d}z = \frac{2\pi i f''(i)}{2!} = 25\pi i e^{5i}$$

since $f''(z) = 25e^{5z}$.

Applications to Harmonic Functions

Suppose that $f(z) = u(z) + iv(z) = u(x,y) + iv(x,y)$ is analytic in a domain D. From Theorem 23.3 we know that all of the derivatives of f are also analytic in D. In particular, this implies that all the partials of u and v of all orders are continuous. This means that we can replace Example 13.9 and Proposition 16.2 with the following.

Theorem 23.7. *If $f(z) = u(z) + iv(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then $u = u(x,y)$ is harmonic in D so that*

$$u_{xx}(x_0, y_0) + u_{yy}(x_0, y_0) = 0$$

for all $(x_0, y_0) \in D$, and $v = v(x,y)$ is harmonic in D so that

$$v_{xx}(x_0, y_0) + v_{yy}(x_0, y_0) = 0$$

for all $(x_0, y_0) \in D$.

Proof. Since $f(z) = u(z) + iv(z)$ is analytic at $z_0 \in D$, we know

$$f'(z_0) = u_x(z_0) + iv_x(z_0) = v_y(z_0) - iu_y(z_0).$$

By Theorem 23.3 we know that all of the derivatives of f are also analytic in D and so the Cauchy-Riemann equations for f' imply

$$f''(z_0) = u_{xx}(z_0) + iv_{xx}(z_0) = v_{yx}(z_0) - iu_{yx}(z_0)$$

and

$$f''(z_0) = v_{xy}(z_0) - iu_{xy}(z_0) = -u_{yy}(z_0) - iy_{yy}(z_0).$$

In particular, we obtain $u_{xy} = u_{yx}$, $v_{xy} = v_{yx}$, and

$$u_{xx}(z_0) + u_{yy}(z_0) = 0 \quad \text{and} \quad v_{xx}(z_0) + v_{yy}(z_0) = 0$$

so that u and v are harmonic at $z_0 \in D$ as required. □

Suppose that $C = C_R = \{|z| = R\}$ is the circle of radius $R > 0$ centred at 0 oriented counterclockwise. We know that if $f(z)$ is analytic inside C_R, then

$$f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

for $|z| < R$. Moreover, we know that

$$u(x,y) = u(z) = \mathrm{Re}(f(z))$$

is harmonic inside C_R. Thus, we want to determine an expression for

$$\mathrm{Re}(f(z)) = \mathrm{Re}\left(\frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta\right).$$

The trick to doing so is to consider the function

$$g(\zeta) = \frac{f(\zeta)\bar{z}}{R^2 - \zeta\bar{z}}$$

which is an analytic function of ζ inside and on C_R. (Note that the denominator, as a function of ζ, is never 0. Why?) Hence, by the Cauchy Integral Theorem,

$$\int_{C_R} g(\zeta)\,\mathrm{d}\zeta = 0 \quad \text{or, equivalently,} \quad \frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)\bar{z}}{R^2 - \zeta\bar{z}}\,\mathrm{d}\zeta = 0.$$

Therefore, if we add this 0 to $f(z)$ we obtain,

$$\begin{aligned} f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + \frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)\bar{z}}{R^2 - \zeta\bar{z}}\,\mathrm{d}\zeta &= \frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)}{\zeta - z} + \frac{f(\zeta)\bar{z}}{R^2 - \zeta\bar{z}}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\int_{C_R} \left[\frac{1}{\zeta - z} + \frac{\bar{z}}{R^2 - \zeta\bar{z}}\right] f(\zeta)\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\int_{C_R} \frac{R^2 - |z|^2}{(\zeta - z)(R^2 - \zeta\bar{z})} f(\zeta)\,\mathrm{d}\zeta. \end{aligned}$$

If we now parametrize C_R by $\zeta = Re^{it}$, $0 \le t \le 2\pi$, then we obtain

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{R^2 - |z|^2}{(Re^{it} - z)(R^2 - Re^{it}\bar{z})} f(Re^{it}) \cdot iRe^{it}\,\mathrm{d}t \\
&= \frac{R^2 - |z|^2}{2\pi}\int_0^{2\pi} \frac{f(Re^{it})}{(Re^{it} - z)(Re^{-it} - \bar{z})}\,\mathrm{d}t \\
&= \frac{R^2 - |z|^2}{2\pi}\int_0^{2\pi} \frac{f(Re^{it})}{|Re^{it} - z|^2}\,\mathrm{d}t.
\end{aligned}$$

If we now write the analytic function $f(z)$ as $f(z) = u(z) + iv(z)$, and then write $z = re^{i\theta}$ as the polar form of z, we obtain

$$\mathrm{Re}\left(\frac{f(Re^{it})}{|Re^{it} - z|^2}\right) = \mathrm{Re}\left(\frac{u(Re^{it}) + iv(Re^{it})}{|Re^{it} - re^{i\theta}|^2}\right) = \frac{u(Re^{it})}{R^2 + r^2 - 2rR\cos(t - \theta)}.$$

Thus, we conclude

$$u(z) = u(re^{i\theta}) = \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{u(Re^{it})}{R^2 + r^2 - 2rR\cos(t - \theta)}\,\mathrm{d}t$$

is harmonic for $z = re^{i\theta}$ with $|z| = r < R$.

Example 23.8. As an example, observe that if $z = 0$ and $R = 1$, then

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})\,\mathrm{d}t$$

which is sometimes called the *circumferential mean value theorem*. This result also has a probabilistic interpretation. The value of the harmonic function at the centre of the unit disk is the uniform average of the values of that function around the unit circle.

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Lecture #24: Applications to Harmonic Functions

We know from Theorem 23.7 that if a function $f(z)$ is analytic in a domain D, then its real part is harmonic in D. We will now prove a partial converse, namely that if a function u is harmonic in a *simply connected domain* D, then there is an analytic function in D whose real part is u.

Theorem 24.1. *Suppose that D is a simply connected domain. If $u = u(z) = u(x, y)$ is harmonic in D, then there is a function $f(z)$ which is analytic in D with* $\operatorname{Re}(f(z)) = u(z)$.

Proof. Suppose that u is harmonic in D so that, by assumption, the second partials u_{xx}, u_{yy}, and $u_{xy} = u_{yx}$ are continuous in D, and

$$u_{xx}(z_0) + u_{yy}(z_0) = 0$$

for every $z_0 \in D$. Suppose that we now set

$$g(z) = u_x(z) - iu_y(z).$$

Observe that g satisfies the Cauchy-Riemann equations in D; that is,

$$\frac{\partial}{\partial x}\operatorname{Re}(g(z)) = \frac{\partial}{\partial x}u_x(z) = u_{xx}(z) = -u_{yy}(z) = \frac{\partial}{\partial y}(-u_y(z)) = \frac{\partial}{\partial y}\operatorname{Im}(g(z))$$

using the fact that $u_{xx} + u_{yy} = 0$ and

$$\frac{\partial}{\partial y}\operatorname{Re}(g(z)) = \frac{\partial}{\partial y}u_x(z) = u_{xy}(z) = u_{yx}(z) = \frac{\partial}{\partial x}(u_y(z)) = -\frac{\partial}{\partial x}\operatorname{Im}(g(z))$$

using the fact that $u_{xy} = u_{yx}$. Since the partials of u are continuous, we conclude from Theorem 13.8 that $g(z)$ is analytic in D. Therefore, we conclude there exists an analytic function $G(z)$ such that $G'(z) = g(z)$. If we write $G(z) = \varphi(z) + i\psi(z)$, then $G(z)$ satisfies the Cauchy-Riemann equations (since $G(z)$ is analytic) so that $\varphi_x(z) = \psi_y(z)$ and $\varphi_y(z) = -\psi_x(z)$. Moreover,

$$G'(z) = \varphi_x(z) + i\psi_x(z) = \varphi_x(z) - i\varphi_y(z)$$

so that $G'(z) = g(z)$ implies

$$\varphi_x(z) - i\varphi_y(z) = u_x(z) - iu_y(z).$$

That is, $u_x(z) = \varphi_x(z)$ and $u_y(z) = \varphi_y(z)$. This means that $u - \varphi = c$ for some real constant c. Hence, the required analytic function $f(z)$ is

$$f(z) = G(z) + c$$

and the proof is complete. □

We can now prove a very interesting property of harmonic functions known as the Maximum Principle. Suppose that $u = u(z) = u(x, y)$ is harmonic in a simply connected domain D. Let $f(z) = u(z) + iv(z)$ be an analytic function in D whose existence is guaranteed by Theorem 24.1. If we now consider the function $e^{f(z)}$, then we observe that

$$|e^{f(z)}| = |e^{u(z)+iv(z)}| = |e^{u(z)}||e^{iv(z)}| = e^{u(z)}$$

since $|e^{iv(z)}| = 1$ and $e^{u(z)} > 0$. The fact that the exponential is a monotonically increasing function of a real variable implies that the maximum points of $u(z)$ must coincide with the maximum points of the modulus of the analytic function $e^{f(z)}$.

Theorem 24.2 (Maximum Principle for Harmonic Functions)**.** *If $u = u(x, y)$ is harmonic in a simply connected domain D and $u(z)$ achieves its maximum value at some point z_0 in D, then $u(z)$ is constant in D.*

To state the theorem in slightly different language, a harmonic function $u(z)$ cannot achieve its maximum at an interior point $z_0 \in D$ unless $u(z)$ is constant.

Of course, the minimum points of $u(z)$ are just the maximum points of $-u(z)$. This means that we also have a minimum principle for harmonic functions.

Theorem 24.3 (Minimum Principle for Harmonic Functions)**.** *If $u = u(x, y)$ is harmonic in a simply connected domain D and $u(z)$ achieves its minimum value at some point z_0 in D, then $u(z)$ is constant in D.*

Combining these results, we arrive at the following theorem.

Theorem 24.4. *A function $u(x, y)$ that is harmonic in a bounded simply connected domain and continuous up to and including the boundary attains its maximum and minimum values on the boundary.*

Supplement: The Dirichlet Problem

Let D be a domain and suppose that $g(z)$ for $z \in C = \partial D$ is a given continuous function. The *Dirichlet Problem* for D is to find a function $u(z) = u(x, y)$ that is continuous on $\overline{D} = D \cup C$, harmonic in D, and satisfies $u(z) = g(z)$ for $z \in C$.

In the case when D is the simply connected domain $D = \{|z| < R\}$, we can solve the Dirichlet Problem. We know from last lecture that

$$u(z) = u(re^{i\theta}) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u(Re^{it})}{R^2 + r^2 - 2rR\cos(t-\theta)}\,\mathrm{d}t \tag{$*$}$$

is harmonic for $z = re^{i\theta}$ with $|z| = r < R$. We now conclude from Theorem 24.1 that if $u(z)$ is harmonic in $D = \{|z| < R\}$, then $u(z)$ can be represented by $(*)$.

Example 24.5. Show

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(t-\theta)}\,\mathrm{d}t = 1.$$

Solution. The function $u(z) = 1$ is clearly harmonic in $D = \{|z| < R\}$ so substituting $u = 1$ into $(*)$ yields

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(t-\theta)}\,\mathrm{d}t = 1.$$

If we also want $u(z) = g(z)$ for $z \in C_R = \partial D$, or equivalently, $u(Re^{it}) = g(Re^{it})$, $0 \le t \le 2\pi$, then we guess that

$$u(z) = \begin{cases} \dfrac{R^2 - r^2}{2\pi} \displaystyle\int_0^{2\pi} \frac{g(Re^{it})}{R^2 + r^2 - 2rR\cos(t-\theta)}\,\mathrm{d}t, & \text{for } z = re^{i\theta} \text{ with } |z| = r < R, \\ g(z), & \text{for } z = Re^{i\theta}, \end{cases}$$

is the desired function.

Note that $u(z)$ has the properties that $u(z)$ is harmonic in D, continuous in D, and satisfies $u(z) = g(z)$ for $z \in C$. By assumption, $g(z)$ is continuous on C. This means that $u(z)$ is continuous both in D and on C. However, we *do not* know that $u(z)$ is continuous on $\overline{D} = D \cup C$ since we have not verified that

$$\lim_{z_0 \to z,\ |z_0| < |z| = R} u(z_0) = g(z),$$

or, equivalently, that

$$\lim_{r \uparrow R} \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{g(Re^{it})}{R^2 + r^2 - 2rR\cos(t-\theta)}\,\mathrm{d}t = g(Re^{i\theta}).$$

Without loss of generality, assume that $R = 1$ and $\theta = 0$ so that we must show

$$\lim_{r \uparrow 1} \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{g(e^{it})}{1 + r^2 - 2r\cos t}\,\mathrm{d}t = g(1).$$

Unfortunately, a completely rigorous proof of this fact is beyond the scope of Math 312. We can, however, give the correct intuition for why it is true. Observe that

$$\frac{1}{2\pi}\lim_{r\uparrow 1}\frac{1-r^2}{1+r^2-2r\cos t}$$

depends on the value of $t\in[0,2\pi]$. There are two possibilities: (i) if $t=0$ or $t=2\pi$, then $\cos t=1$ so that

$$\frac{1}{2\pi}\lim_{r\uparrow 1}\frac{1-r^2}{1+r^2-2r}=\infty,$$

and (ii) if $0<t<2\pi$, then $|\cos t|<1$ so that $1+r^2-2r\cos t\neq 0$ for r sufficiently close to 1 implying that

$$\frac{1}{2\pi}\lim_{r\uparrow 1}\frac{1-r^2}{1+r^2-2r\cos t}=0.$$

The Dirac delta "function" is sometimes used to describe this limit. Note that the Dirac delta function is not a function in the usual sense but rather a *generalized function* or *tempered distributions*; as such, the following formulas, though suggestive, are not meaningful mathematically. Let $\delta_0(t)$ have the properties that

$$\delta_0(t)=\begin{cases}0, & t\neq 0,\\ +\infty, & t=0,\end{cases}$$

and

$$\int_{-\infty}^{\infty}\delta_0(t)\,\mathrm{d}t=1. \tag{$\dagger$}$$

That is, we have

$$\frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos t}\to\delta_0(t)$$

for $0\le t<2\pi$ as $r\uparrow 1$. Note that, as a result of Example 24.5, the factor of 2π is necessary for ($\dagger$) to hold. One useful identity involving the Dirac delta function is that if $h(t)$, $t\in\mathbb{R}$, is a real-valued function, then

$$\int_{-\infty}^{\infty}\delta_0(t)h(t)\,\mathrm{d}t=h(0).$$

Thus, assuming that we can interchange limits and integrals, we arrive at

$$\begin{aligned}
\lim_{r\uparrow 1}\frac{1-r^2}{2\pi}\int_0^{2\pi}\frac{g(e^{it})}{1+r^2-2r\cos t}\,\mathrm{d}t &= \lim_{r\uparrow 1}\int_0^{2\pi}\frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos t}g(e^{it})\,\mathrm{d}t\\
&=\int_0^{2\pi}\lim_{r\uparrow 1}\left[\frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos t}g(e^{it})\right]\,\mathrm{d}t\\
&=\int_0^{2\pi}g(e^{it})\left[\lim_{r\uparrow 1}\frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos t}\right]\,\mathrm{d}t\\
&=\int_0^{2\pi}g(e^{it})\delta_0(t)\,\mathrm{d}t\\
&=g(e^{i0})=g(1).
\end{aligned}$$

as required.

Finally, we should note that the *integrating kernel*

$$\frac{1}{2\pi}\frac{R^2-r^2}{R^2+r^2-2rR\cos(t-\theta)}$$

has a special name. It is the *Poisson kernel*; that is, if $z = re^{i\theta} \in D$ and $w = Re^{it} \in C = \partial D$, then

$$P(w;z) = \frac{1}{2\pi}\frac{|w|^2-|z|^2}{|w-z|^2} \quad \text{or, equivalently,} \quad P(R,t;r,\theta) = \frac{1}{2\pi}\frac{R^2-r^2}{R^2+r^2-2rR\cos(t-\theta)}.$$

Thus, the solution to the Dirichlet problem for $\{|z| < R\}$ subject to boundary conditions $g(Re^{it})$ can be expressed as

$$u(re^{i\theta}) = \int_0^{2\pi} P(R,t;r,\theta)g(Re^{it})\,\mathrm{d}t.$$

This representation of $u(z)$ is often called *Poisson's integral formula.*

Remark. Note that we have introduced the Poisson kernel from a purely analytic point-of-view. However, as a result of Exercise 24.5 we can view the Poisson kernel as a probability density function on the boundary of the disk of radius R. That is, $P(R,t;r,\theta) \geq 0$ and satisfies

$$\int_0^{2\pi} P(R,t;r,\theta)\,\mathrm{d}t = 1.$$

Moreover, it turns out that Brownian motion and the Poisson kernel are intimately connected. The density of the first exit from the disk of radius R by two dimensional Brownian motion started at the interior point $z = re^{i\theta}$ is exactly $P(R,t;r,\theta)$. Although the Poisson kernel has been studied for over 150 years, and the relationship between Brownian motion and the Poisson kernel has been understood for over 60 years, the Poisson kernel is still vital for modern mathematics. In fact, the Poisson kernel plays a significant role in the Fields Medal winning work of Wendelin Werner (2006) and Stas Smirnov (2010) on the Schramm-Loewner evolution.

Lecture #25: Taylor Series

Our primary goal for today is to prove that if $f(z)$ is an analytic function in a domain D, then $f(z)$ can be expanded in a Taylor series about any point $a \in D$. Moreover, the Taylor series for $f(z)$ converges uniformly to $f(z)$ for any z in a closed disk centred at a and contained entirely in D.

Theorem 25.1. *Suppose that $f(z)$ is analytic in the disk $\{|z-a| < R\}$. Then the sequence of Taylor polynomials for $f(z)$ about the point a, namely*

$$T_n(z; f, a) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(z-a)^j,$$

converges to $f(z)$ for all z in this disk. Furthermore, the convergence is uniform in any closed subdisk $\{|z-a| \le R' < R\}$. In particular, if $f(z)$ is analytic in $\{|z-a| < R\}$, then

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!}(z-a)^j. \tag{1}$$

We call (1) the *Taylor series for $f(z)$ about the point a.*

Proof. It is sufficient to prove uniform convergence in every subdisk $\{|z-a| \le R' < R\}$. Set $R'' = (R+R')/2$ and consider the closed contour $C = \{|z-a| = R''\}$ oriented counterclockwise. By the Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta. \tag{$*$}$$

Observe that

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta-a)-(z-a)} = \frac{1}{\zeta-a}\frac{1}{1-\left(\frac{z-a}{\zeta-a}\right)} = \frac{1}{\zeta-a}\frac{1}{1-w} \quad \text{where} \quad w = \left(\frac{z-a}{\zeta-a}\right)$$

and so using the fact that

$$\frac{1-w^{n+1}}{1-w} = 1 + w + w^2 + \cdots + w^n \quad \text{or equivalently} \quad \frac{1}{1-w} = 1 + w + \cdots + w^n + \frac{w^{n+1}}{1-w},$$

we conclude

$$\begin{aligned}\frac{1}{1-\left(\frac{z-a}{\zeta-a}\right)} &= 1 + \left(\frac{z-a}{\zeta-a}\right) + \cdots + \left(\frac{z-a}{\zeta-a}\right)^n + \frac{\left(\frac{z-a}{\zeta-a}\right)^{n+1}}{1-\left(\frac{z-a}{\zeta-a}\right)} \\ &= 1 + \left(\frac{z-a}{\zeta-a}\right) + \cdots + \left(\frac{z-a}{\zeta-a}\right)^n + \frac{\zeta-a}{\zeta-z}\left(\frac{z-a}{\zeta-a}\right)^{n+1}\end{aligned}$$

and

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a}\left[1 + \left(\frac{z-a}{\zeta - a}\right) + \cdots + \left(\frac{z-a}{\zeta-a}\right)^n + \frac{\zeta - a}{\zeta - z}\left(\frac{z-a}{\zeta-a}\right)^{n+1}\right]. \qquad (**)$$

Substituting $(**)$ into $(*)$ we conclude

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - a}\left[1 + \left(\frac{z-a}{\zeta - a}\right) + \cdots + \left(\frac{z-a}{\zeta-a}\right)^n + \frac{\zeta - a}{\zeta - z}\left(\frac{z-a}{\zeta-a}\right)^{n+1}\right] \mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - a}\,\mathrm{d}\zeta + \frac{(z-a)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-a)^2}\,\mathrm{d}\zeta + \cdots + \frac{(z-a)^n}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,\mathrm{d}\zeta \\
&\qquad + \frac{(z-a)^{n+1}}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)(\zeta-a)^{n+1}}\,\mathrm{d}\zeta.
\end{aligned}$$

However, from the Cauchy Integral Formula, we know

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - a}\,\mathrm{d}\zeta = f(a), \quad \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - a)^2}\,\mathrm{d}\zeta = f'(a), \quad \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - a)^3}\,\mathrm{d}\zeta = \frac{f''(a)}{2!},$$

and in general

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - a)^{j+1}}\,\mathrm{d}\zeta = \frac{f^{(j)}(a)}{j!}$$

so that

$$\begin{aligned}
f(z) &= f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \frac{(z-a)^{n+1}}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)(\zeta-a)^{n+1}}\,\mathrm{d}\zeta \\
&= T_n(z; f, a) + R_n(z; f, a).
\end{aligned}$$

Thus, we see that in order to show that $T_n(z; f, a)$ converges to $f(z)$ uniformly for $|z-a| \le R'$, it suffices to show that $R_n(z; f, a)$ converges to 0 uniformly for $|z - a| \le R'$. Suppose, therefore, that $|z - a| \le R'$ and $|\zeta - a| = R''$ where $R'' = (R + R')/2$ as before. By the triangle inequality,

$$|\zeta - z| \ge R'' - R' = \frac{R + R'}{2} - R' = \frac{R - R'}{2},$$

and so

$$\begin{aligned}
|R_n(z; f, a)| &= \left|\frac{(z-a)^{n+1}}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)(\zeta-a)^{n+1}}\,\mathrm{d}\zeta\right| \le \frac{1}{2\pi}\int_C \left|\frac{f(\zeta)(z-a)^{n+1}}{(\zeta - z)(\zeta-a)^{n+1}}\right| \mathrm{d}\zeta \\
&\le \frac{1}{2\pi}\int_C \frac{|f(\zeta)|(R')^{n+1}}{(R'')^{n+1}(R-R')/2}\,\mathrm{d}\zeta \\
&\le \frac{1}{\pi}\max_{\zeta \in C}|f(\zeta)|\left(\frac{R'}{R''}\right)^{n+1}\frac{1}{R - R'}\;\ell(C)
\end{aligned}$$

where $\ell(C) = 2\pi R''$ is the arclength of C. That is, after some simplification, we obtain

$$|R_n(z; f, a)| \le \left(\frac{2R'}{R + R'}\right)^n \frac{2R'}{R - R'}\max_{\zeta \in C}|f(\zeta)|.$$

Notice that the right side of the previous inequality is independent of z. Since $2R' < R + R'$, the right side can be made less than any $\epsilon > 0$ by taking n sufficiently large. This gives the required uniform convergence. $\square$

Example 25.2. Find the Taylor series for $f(z) = e^z$ about $a = 0$.

Solution. Since $f^{(n)}(z) = e^z$ so that $f^{(n)}(0) = 1$ for all non-negative integers n, we conclude

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{z^j}{j!}$$

for every $z \in \mathbb{C}$.

Example 25.3. Find the Taylor series for both $f_1(z) = \sin z$ and $f_2(z) = \cos z$ about $a = 0$, and then show that the Taylor series for e^{iz} equals the sum of the Taylor series for $\cos z$ and $i \sin z$.

Solution. Observe that $f_1'(z) = \cos z = f_2(z)$ and $f_2'(z) = -\sin z = -f_1(z)$. Since $\cos 0 = 1$ and $\sin 0 = 0$, we obtain

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!}$$

and

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{(2j)!}$$

for every $z \in \mathbb{C}$. Observe that

$$\begin{aligned}
e^{iz} &= 1 + (iz) + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \frac{(iz)^6}{6!} + \cdots \\
&= \left(1 + \frac{(iz)^2}{2!} + \frac{(iz)^4}{4!} + \frac{(iz)^6}{6!} + \cdots\right) + \left(iz + \frac{(iz)^3}{3!} + \frac{(iz)^5}{5!} + \frac{(iz)^7}{7!} + \cdots\right) \\
&= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) + i\left(z + \frac{i^2 z^3}{3!} + \frac{i^4 z^5}{5!} + \frac{i^6 z^7}{7!} + \cdots\right) \\
&= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) \\
&= \cos z + i \sin z
\end{aligned}$$

as expected. It is worth noting that term-by-term manipulations of the sum of Taylor series are justified by Theorem 25.1 since the Taylor series involved converge uniformly in closed disks about the point $a = 0$.

Remark. Sometimes the phrase *Maclaurin series* is used in place of Taylor series when $a = 0$.

Theorem 25.4. *If $f(z)$ is analytic at z_0, then the Taylor series for $f'(z)$ at z_0 can be obtained by termwise differentiation of the Taylor series for $f(z)$ about z_0 and converges in the same disk as the Taylor series for $f(z)$.*

Proof. Since $f(z)$ is analytic at z_0, the Taylor series for $f(z)$ about z_0 is given by

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z-z_0)^j.$$

By termwise differentiation, we obtain

$$f'(z) = \sum_{j=0}^{\infty} j\frac{f^{(j)}(z_0)}{j!}(z-z_0)^{j-1} = \sum_{j=1}^{\infty} \frac{f^{(j)}(z_0)}{(j-1)!}(z-z_0)^{j-1}. \tag{$*$}$$

Suppose now that $g(z) = f'(z)$. By Theorem 23.3, we know that $g(z)$ is analytic at z_0 so that its Taylor series is

$$g(z) = \sum_{j=0}^{\infty} \frac{g^{(j)}(z_0)}{j!}(z-z_0)^j.$$

However, $g^{(j)}(z_0) = f^{(j+1)}(z_0)$ so that

$$f'(z) = g(z) = \sum_{j=0}^{\infty} \frac{g^{(j)}(z_0)}{j!}(z-z_0)^j = \sum_{j=0}^{\infty} \frac{f^{(j+1)}(z_0)}{j!}(z-z_0)^j. \tag{$**$}$$

But by a change of index, it is clear that $(*)$ and $(**)$ are equal as required. $\square$

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Prof. Michael Kozdron

Lecture #26: Taylor Series and Isolated Singularities

Recall from last class that if $f(z)$ is analytic at a, then

$$f(z) = \sum_{j=0}^{\infty} \frac{f^j(a)}{j!}(z-a)^j$$

for all z in some neighbourhood of a. This neighbourhood, denoted by $\{|z-a| < R\}$, is called the *disk of convergence of the Taylor series for* $f(z)$.

Example 26.1. We know from Lecture #5 that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

for $|z| < 1$. Since $(1-z)^{-1}$ is analytic for $|z| < 1$, we conclude this must be its Taylor series expansion about $a = 0$. Moreover, since $|-z^2| < 1$ whenever $|z| < 1$, we find

$$\frac{1}{1+z^2} = 1 + (-z^2) + (-z^2)^2 + (-z^2)^3 + \cdots = 1 - z^2 + z^4 - z^6 + z^8 - z^{10} + \cdots$$

for $|z| < 1$. Observe now that

$$\frac{\mathrm{d}}{\mathrm{d}z} \arctan z = \frac{1}{1+z^2}.$$

Consequently we can use Theorem 25.4, which tells us that the derivative of an analytic function $f(z)$ can be obtained by termwise differentiation of the Taylor series of $f(z)$, to conclude that the Taylor series expansion about 0 for $\arctan z$ must be

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

for $|z| < 1$ since

$$\frac{\mathrm{d}}{\mathrm{d}z} \arctan z = \frac{\mathrm{d}}{\mathrm{d}z}\left(z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots\right) = 1 - z^2 + z^4 - z^6 + z^8 - z^{10} + \cdots = \frac{1}{1+z^2}.$$

We can now recover the famous Leibniz formula for π from 1682; that is, since $\arctan(1) = \pi/4$, we find

$$\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

or equivalently,

$$\pi = 4\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}.$$

Observe that in the last example we obtained the Taylor series for

$$\frac{1}{1+z^2}$$

by formally plugging $-z^2$ into the Taylor series for

$$\frac{1}{1-z}.$$

Suppose that we try do the same thing with a point at which the Taylor series is not analytic. For instance, we know that

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

for all $z \in \mathbb{C}$. However, is it true that

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2!} + \frac{1}{3!z^3} + \cdots ,$$

at least for $z \neq 0$? Our goal is to now develop the theory of Laurent series which will provide us with the means to understand the series expansion for $e^{1/z}$ given above.

Consider again the function $e^{1/z}$. Observe that the point $z = 0$ is special because it is the only point at which $e^{1/z}$ fails to be analytic. We call such a point an *isolated singularity.*

Definition. A point z_0 is called an *isolated singular point* of $f(z)$ if $f(z)$ is not analytic at z_0 but is analytic at all points in some small neighbourhood of z_0.

Example 26.2. If $f(z) = e^{1/z}$, then $z = 0$ is an isolated singular point of $f(z)$.

Example 26.3. If

$$f(z) = \frac{1}{z},$$

then $z_0 = 0$ is an isolated singular point of $f(z)$.

Example 26.4. If

$$f(z) = \csc z = \frac{1}{\sin z},$$

then $f(z)$ has isolated singular points at $z = n\pi$ for $n \in \mathbb{Z}$.

Example 26.5. If

$$f(z) = \frac{1}{\sin(1/z)},$$

then $f(z)$ has isolated singular points at those points for which $\sin(1/z) = 0$, namely

$$\frac{1}{z} = n\pi \quad \text{or equivalently} \quad z = \frac{1}{n\pi}$$

for $n = \pm 1, \pm 2, \pm 3, \ldots$. Note that $z = 0$ is not an isolated singular point since there are points of the form $1/(n\pi)$ arbitrarily close to 0 for n sufficiently large. In other words, 0 is a cluster point or an accumulation point of the sequence of isolated singular points $1/(n\pi)$, $n = 1, -1, 2, -2, 3, -3, \ldots$

The basic idea is as follows. For a function $f(z)$ analytic for $|z - z_0| < R$, we have

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j.$$

However, if a function is analytic in an annulus only, say $r < |z - z_0| < R$ (with $r = 0$ and $R = \infty$ allowed), then our expansion will have negative powers of $(z - z_0)$; for example,

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2!} + \frac{1}{3!z^3} + \cdots .$$

Example 26.6. Determine a series expansion for

$$f(z) = \frac{1 + 2z}{z^2 + z^3}$$

in powers of z.

Solution. Recall that

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots .$$

Hence,

$$\begin{aligned}
f(z) = \frac{1+2z}{z^2(1+z)} = \frac{1+2z}{z^2}\frac{1}{1+z} &= \frac{1+2z}{z^2}\left(1 - z + z^2 - z^3 + \cdots\right) \\
&= \frac{1}{z^2}\left[\left(1 - z + z^2 - z^3 + \cdots\right) + 2z\left(1 - z + z^2 - z^3 + \cdots\right)\right] \\
&= \frac{1}{z^2}\left(1 + z - z^2 + z^3 - z^4 + \cdots\right) \\
&= \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots .
\end{aligned}$$

This expansion is the so-called Laurent series expansion of $f(z)$ about $z_0 = 0$. The next several lectures will be devoted to the development of the theory of Laurent series. Here is one important use of the series expansion just determined.

Example 26.7. Suppose that $C = \{|z| = 1/2\}$ oriented counterclockwise. Compute

$$\int_C \frac{1+2z}{z^2 + z^3}\, \mathrm{d}z.$$

Solution. Assuming that

$$\frac{1+2z}{z^2+z^3} = \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots ,$$

we obtain

$$\begin{aligned}
\int_C \frac{1+2z}{z^2+z^3}\, \mathrm{d}z &= \int_C \left(\frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots\right) \mathrm{d}z \\
&= \int_C \frac{1}{z^2}\, \mathrm{d}z + \int_C \frac{1}{z}\, \mathrm{d}z - \int_C 1\, \mathrm{d}z + \int_C z\, \mathrm{d}z - \int_C z^2\, \mathrm{d}z + \cdots \\
&= 0 + 2\pi i + 0 + 0 + 0 + \cdots \\
&= 2\pi i.
\end{aligned}$$

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Lecture #27: Laurent Series

Recall from Lecture #26 that we considered the function

$$f(z) = \frac{1+2z}{z^2+z^3}$$

and we formally manipulated $f(z)$ to obtain the infinite expansion

$$f(z) = \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots .$$

Observe that $f(z)$ is analytic in the annulus $0 < |z| < 1$. Does

$$\frac{1+2z}{z^2+z^3} = \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots$$

for all $0 < |z| < 1$? The answer turns out to be yes. Thus, our goal for today is to prove that if a function $f(z)$ is analytic in an annulus, then it has an infinite series expansion which converges for all z in the annulus. This expansion is known as the *Laurent series for* $f(z)$.

Theorem 27.1. *Suppose that $f(z)$ is analytic in the annulus $r < |z - z_0| < R$ (with $r = 0$ and $R = \infty$ allowed). Then $f(z)$ can be represented as*

$$f(z) = \sum_{j=0}^{\infty} a_j (z-z_0)^j + \sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j} \qquad (*)$$

where

$$a_j = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\,\mathrm{d}\zeta, \quad j = 0, \pm 1, \pm 2, \ldots,$$

and C is any closed contour oriented counterclockwise that surrounds z_0 and lies entirely in the annulus.

The proof is very similar to the proof of Theorem 25.1 for the Taylor series representation of an analytic function in a disk $|z-z_0| < R$. We will not include the full proof, but instead give an indication of where the formula for a_j comes from. Suppose that $f(z)$ can be represented as

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

with convergence in the annulus $r < |z - z_0| < R$. Observe that

$$\frac{1}{2\pi i} f(z)(z-z_0)^{-j} = \sum_{k=-\infty}^{\infty} \frac{a_k}{2\pi i}(z-z_0)^{k-j}$$

and so

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^j}\,\mathrm{d}z = \sum_{k=-\infty}^{\infty} \frac{a_k}{2\pi i}\int_C \frac{1}{(z-z_0)^{j-k}}\,\mathrm{d}z$$

where C is any closed contour oriented counterclockwise that surrounds z_0 and lies entirely in the annulus. We now observe from Theorem 21.1 that

$$\int_C \frac{1}{(z-z_0)^{j-k}}\,\mathrm{d}z = 2\pi i \quad \text{if} \quad k = j-1$$

and

$$\int_C \frac{1}{(z-z_0)^{j-k}}\,\mathrm{d}z = 0 \quad \text{if} \quad k \neq j-1$$

so that

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^j}\,\mathrm{d}z = \sum_{k=-\infty}^{\infty} \frac{a_k}{2\pi i}\int_C \frac{1}{(z-z_0)^{j-k}}\,\mathrm{d}z = a_{j-1}.$$

In other words,

$$a_{j-1} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^j}\,\mathrm{d}z \quad \text{or, equivalently,} \quad a_j = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{j+1}}\,\mathrm{d}z.$$

Remark. Observe that

$$a_j = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z_0)^{j+1}}\,\mathrm{d}\zeta$$

so, at least for $j = 0, 1, 2, \ldots$, it might be tempting to use the Cauchy Integral Formula (Theorem 23.4) to try and conclude that a_j is equal to

$$\frac{f^{(j)}(z_0)}{j!}$$

as was the case in the Taylor series derivation. This is not true, however, since the assumption on $f(z)$ is that it is analytic in the annulus $r < |z - z_0| < R$. This means that if C is a closed contour oriented counterclockwise lying in the annulus and surrounding z_0, there is no guarantee that $f(z)$ is analytic everywhere inside C which is the assumption required in order to apply the Cauchy Integral Formula. Thus, although there is a seemingly simple formula for the coefficients a_j in the Laurent series expansion, the computation of a_j as a contour integral is not necessarily a straightforward application of the Cauchy Integral Formula.

Example 27.2. Suppose that

$$f(z) = \frac{1+2z}{z^2+z^3}$$

which is analytic for $0 < |z| < 1$. Show that the Laurent series expansion of $f(z)$ for $0 < |z| < 1$ is

$$\frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots .$$

Solution. Suppose that C is any closed contour oriented counterclockwise lying entirely in $\{0 < |z| < 1\}$ and surrounding $z_0 = 0$. Consider

$$a_j = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_C \frac{1+2\zeta}{\zeta^2+\zeta^3}\cdot\frac{1}{\zeta^{j+1}}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_C \frac{(1+2\zeta)/(1+\zeta)}{\zeta^{j+3}}\,\mathrm{d}\zeta.$$

The reason for writing it in this form is that now we can apply the Cauchy Integral Formula to compute

$$\frac{1}{2\pi i}\int_C \frac{(1+2\zeta)/(1+\zeta)}{\zeta^{j+3}}\,\mathrm{d}\zeta.$$

Observe that the function

$$g(z) = \frac{1+2z}{1+z}$$

is analytic inside and on C. Thus, the Cauchy Integral Theorem implies that if $j \le -3$, then

$$\frac{1}{2\pi i}\int_C \frac{(1+2\zeta)/(1+\zeta)}{\zeta^{j+3}}\,\mathrm{d}\zeta = 0$$

so that $a_{-3} = a_{-4} = \cdots = 0$. The Cauchy Integral Formula implies that if $j \ge -2$, then

$$\frac{1}{2\pi i}\int_C \frac{(1+2\zeta)/(1+\zeta)}{\zeta^{j+3}}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_C \frac{g(\zeta)}{\zeta^{j+3}}\,\mathrm{d}\zeta = \frac{g^{(j+2)}(0)}{(j+2)!}.$$

Note that if $j = -2$, then $a_{-2} = g(0) = 1$. In order to compute successive derivatives of $g(z)$, observe that

$$g(z) = \frac{1+2z}{1+z} = \frac{1}{1+z} + \frac{2z}{1+z}$$

Now, if $k = 1, 2, 3, \ldots$, then

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\frac{1}{1+z}\right) = (-1)^k\frac{k!}{(1+z)^{k+1}}$$

and

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\frac{z}{1+z}\right) = (-1)^{k+1}\frac{k!}{(1+z)^k} + (-1)^k\frac{k!z}{(1+z)^{k+1}}$$

so that

$$g^{(k)}(0) = (-1)^k k! + 2(-1)^{k+1}k! = (-1)^{k+1}k! \quad \text{for } k = 1, 2, 3, \ldots.$$

This implies

$$a_j = \frac{g^{(j+2)}(0)}{(j+2)!} = \frac{(-1)^{j+3}(j+2)!}{(j+2)!} = (-1)^{j+3} = (-1)^{j+1} \quad \text{for } j = -1, 0, 1, 2\ldots$$

so that the Laurent series expansion of $f(z)$ for $0 < |z| < 1$ is

$$\frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots = \frac{1}{z^2} + \sum_{j=-1}^{\infty}(-1)^{j+1}z^j.$$

Remark. We will soon learn other methods for determining Laurent series expansions that do not require the computation of contour integrals.

A first look at residue theory as an application of Laurent series

One important application of the theory of Laurent series is in the computation of contour integrals. Suppose that $f(z)$ is analytic in the annulus $0 < |z - z_0| < R$ so that $f(z)$ has a Laurent series expansion

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j}.$$

Let C be any closed contour oriented counterclockwise lying entirely in the annulus and surrounding z_0 so that

$$\int_C f(z)\,\mathrm{d}z = \sum_{j=0}^{\infty} a_j \int_C (z - z_0)^j\,\mathrm{d}z + \sum_{j=1}^{\infty} a_{-j} \int_C (z - z_0)^{-j}\,\mathrm{d}z.$$

We know from Theorem 21.1 that

$$\int_C (z - z_0)^j\,\mathrm{d}z = \begin{cases} 2\pi i, & \text{if } j = -1, \\ 0, & \text{if } j \neq -1, \end{cases}$$

so that

$$\int_C f(z)\,\mathrm{d}z = 2\pi i a_{-1}.$$

Thus, we see that the coefficient a_{-1} in the Laurent series expansion of $f(z)$ in an annulus of the form $0 < |z - z_0| < R$ is of particular importance. In fact, it has a name and is called the *residue of* $f(z)$ *at* z_0, denoted by $a_{-1} = \operatorname{Res}(f; z_0)$.

Example 27.3. Since the Laurent series of

$$f(z) = \frac{1 + 2z}{z^2 + z^3}$$

for $0 < |z| < 1$ is

$$\frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots,$$

we conclude that

$$\int_C \frac{1 + 2z}{z^2 + z^3}\,\mathrm{d}z = 2\pi i a_{-1} = 2\pi i$$

for any contour C lying entirely in $\{0 < |z| < 1\}$ and surrounding 0.

Mathematics 312 (Fall 2012) November 19, 2012
Prof. Michael Kozdron

Lecture #28: Calculating Laurent Series

Example 28.1. Determine the Laurent series for

$$f(z) = \frac{1}{(z-1)(2-z)}$$

for **(i)** $1 < |z| < 2$, and **(ii)** $|z| > 2$.

Solution. Note that the only singular points of $f(z)$ occur at 1 and 2. This means that (i) $f(z)$ is analytic in the annulus $1 < |z| < 2$, and (ii) $f(z)$ is analytic for $|z| > 2$. Hence, in either case, the Laurent series for $f(z)$ will necessarily be of the form

$$f(z) = \sum_{j=1}^{\infty} a_j z^j + \sum_{j=1}^{\infty} a_{-j} z^{-j},$$

and so the idea is that we will find the coefficients a_j directly rather than by contour integration. (It is worth stressing that the coefficients in the Laurent series expansions for (i) and (ii) will not necessarily be the same.)

(i) Using partial fractions, we find

$$f(z) = \frac{1}{(z-1)(2-z)} = \frac{1}{z-1} - \frac{1}{z-2}.$$

Now consider

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{j=0}^{\infty} z^j.$$

We know this series converges for $|z| < 1$. However, we are interested in determining a series which converges for $|z| > 1$. Thus, we write

$$\frac{1}{z-1} = \frac{1/z}{1-1/z}$$

and observe that the series

$$\frac{1}{1-1/z} = \sum_{j=0}^{\infty} (1/z)^j$$

converges for $|1/z| < 1$, or equivalently, $|z| > 1$. This implies

$$\frac{1}{z-1} = \frac{1/z}{1-1/z} = \frac{1}{z}\sum_{j=0}^{\infty} z^{-j} = \sum_{j=0}^{\infty} z^{-j-1} = \sum_{j=1}^{\infty} z^{-j} \quad \text{for } |z| > 1.$$

Now consider

$$-\frac{1}{z-2} = \frac{1/2}{1-z/2} = \frac{1}{2}\sum_{j=0}^{\infty} (z/2)^j = \sum_{j=0}^{\infty} \frac{z^j}{2^{j+1}}$$

which converges for $|z/2| < 1$, or equivalently, $|z| < 2$. Thus, when we add these two series, we obtain

$$\frac{1}{z-1} - \frac{1}{z-2} = \sum_{j=1}^{\infty} z^{-j} + \sum_{j=0}^{\infty} \frac{z^j}{2^{j+1}}.$$

Note that the first series converges for $|z| > 1$ while the second series converges for $|z| < 2$. This means that they BOTH converge when $1 < |z| < 2$, and so we have found the Laurent series for $f(z)$ for $1 < |z| < 2$, namely

$$f(z) = \frac{1}{(z-1)(2-z)} = \sum_{j=0}^{\infty} \frac{z^j}{2^{j+1}} + \sum_{j=1}^{\infty} z^{-j}.$$

(ii) We now want the Laurent series for $f(z)$ to converge for $|z| > 2$. We already know that

$$\frac{1}{z-1} = \sum_{j=1}^{\infty} z^{-j}$$

converges for $|z| > 1$. However, the series that we found for

$$\frac{1}{z-2}$$

converges for $|z| < 2$. This means that we need to manipulate it differently, say

$$\frac{1}{z-2} = \frac{1/z}{1-2/z} = \frac{1}{z}\sum_{j=0}^{\infty}(2/z)^j = \sum_{j=0}^{\infty} 2^j z^{-j-1} = \sum_{j=1}^{\infty} 2^{j-1} z^{-j}$$

which converges for $|2/z| < 1$, or equivalently, $|z| > 2$. Thus,

$$f(z) = \frac{1}{(z-1)(2-z)} = \frac{1}{z-1} - \frac{1}{z-2} = \sum_{j=1}^{\infty} z^{-j} - \sum_{j=1}^{\infty} 2^{j-1} z^{-j} = \sum_{j=1}^{\infty} (1-2^{j-1}) z^{-j}$$

for $|z| > 2$.

Example 28.2. Determine the Laurent series for

$$f(z) = \frac{e^{2z}}{(z-1)^3}$$

for all $|z-1| > 0$.

Solution. Observe that $f(z)$ is analytic for $|z-1| > 0$. Now

$$f(z) = \frac{e^{2z}}{(z-1)^3} = \frac{e^{2(z-1+1)}}{(z-1)^3} = e^2 \frac{e^{2(z-1)}}{(z-1)^3}.$$

Recall that the Taylor series for e^w is

$$e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{w^j}{j!}$$

which converges for all $w \in \mathbb{C}$. This implies

$$e^{2(z-1)} = 1 + 2(z-1) + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{2^j(z-1)^j}{j!}.$$

Hence,

$$\frac{e^{2(z-1)}}{(z-1)^3} = (z-1)^{-3} \sum_{j=0}^{\infty} \frac{2^j(z-1)^j}{j!} = \sum_{j=0}^{\infty} \frac{2^j(z-1)^{j-3}}{j!}$$

so that the Laurent series of $f(z)$ for $|z-1| > 0$ is

$$f(z) = \frac{e^{2z}}{(z-1)^3} = e^2 \sum_{j=0}^{\infty} \frac{2^j(z-1)^{j-3}}{j!} = e^2 \left(\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{4}{2(z-1)} + \frac{8}{6} + \cdots \right).$$

Example 28.3. Let

$$\sinh z = \frac{e^z - e^{-z}}{2} = -i \sin(iz)$$

and let

$$f(z) = \frac{1}{z^2 \sinh z}.$$

Determine the first few terms of the Laurent series for $f(z)$ in $0 < |z| < \pi$, and then calculate

$$\int_C \frac{1}{z^2 \sinh z} \, \mathrm{d}z$$

where $C = \{|z| = 1\}$ is the unit circle centred at 0 oriented counterclockwise.

Solution. On Assignment #9 you will show that the Taylor series for the entire function $\sinh z$ is

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{j=0}^{\infty} \frac{z^{2j+1}}{(2j+1)!}$$

which converges for all $z \in \mathbb{C}$. Moreover, it is not too difficult to show that $\sinh z = 0$ if and only if $z \in \{0, \pm i\pi, \pm 2i\pi, \ldots\}$. This implies that $f(z)$ is analytic for $0 < |z| < \pi$. Now

$$f(z) = \frac{1}{z^2 \sinh z} = \frac{1}{z^2 \left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \right)} = \frac{1}{z^3} \frac{1}{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots}.$$

Using the identity

$$\frac{1}{1+w} = 1 - w + w^2 - w^3 + \cdots,$$

we obtain

$$\frac{1}{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots} = 1 - \left(\frac{z^2}{3!} + \frac{z^4}{5!} + \cdots \right) + \left(\frac{z^2}{3!} + \frac{z^4}{5!} + \cdots \right)^2 + \cdots = 1 - \frac{z^2}{6} + \frac{7z^4}{360} + \cdots$$

so that

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} \left(1 - \frac{z^2}{6} + \frac{7z^4}{360} + \cdots \right) = \frac{1}{z^3} - \frac{1}{6z} + \frac{7z}{360} + \cdots$$

for $0 < |z| < \pi$. Hence,

$$\int_C \frac{1}{z^2 \sinh z} \, \mathrm{d}z = \int_C \frac{1}{z^3} - \frac{1}{6z} + \frac{7z}{360} + \cdots \, \mathrm{d}z = -\frac{2\pi i}{6} = -\frac{\pi i}{3}.$$

Lecture #29: Laurent Series and Residue Theory

Example 29.1. Determine the Laurent series of

$$f(z) = \frac{z^2 - 2z + 3}{z - 2}$$

for $|z - 1| > 1$.

Solution. Note that $f(z)$ is analytic for $|z-1| > 1$ since the only singularity for $f(z)$ occurs at $z = 2$. Since we are expanding about the point $z_0 = 1$, the Laurent series will necessarily have the form

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j.$$

Therefore, if we want to expand in powers of $z-1$, we need to turn both our numerator and denominator into functions of $z - 1$. Observe that

$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = \frac{1}{z-1}\frac{1}{1-\frac{1}{z-1}} = \frac{1}{z-1}\sum_{j=0}^{\infty}\left(\frac{1}{z-1}\right)^j = \sum_{j=0}^{\infty}(z-1)^{-j-1} = \sum_{j=1}^{\infty}(z-1)^{-j}$$

for $|1/(z-1)| < 1$, or equivalently, $|z-1| > 1$, and

$$z^2 - 2z + 3 = (z-1)^2 + 2.$$

This yields

$$\begin{aligned} f(z) = \left[(z-1)^2 + 2\right]\sum_{j=1}^{\infty}(z-1)^{-j} &= \sum_{j=1}^{\infty}\left(\frac{1}{z-1}\right)^{j-2} + 2\sum_{j=1}^{\infty}\left(\frac{1}{z-1}\right)^j \\ &= (z-1) + 1 + \sum_{j=3}^{\infty}\left(\frac{1}{z-1}\right)^{j-2} + 2\sum_{j=1}^{\infty}\left(\frac{1}{z-1}\right)^j \\ &= (z-1) + 1 + 3\sum_{j=1}^{\infty}\left(\frac{1}{z-1}\right)^j \end{aligned}$$

for $|z-1| > 1$ as the required Laurent series.

Classifying isolated singularities

We will now focus on functions that have an isolated singularity at z_0. Therefore, suppose that $f(z)$ is analytic in the annulus $0 < |z - z_0| < R$ and has an isolated singularity at z_0. Consider its Laurent series expansion

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j}.$$

We call the part with the negative powers of $(z - z_0)$, namely

$$\sum_{j=1}^{\infty} a_{-j}(z - z_0)^{-j},$$

the *principal part* of the Laurent series. There are three mutually exclusive possibilities for the principal part.

(i) If $a_j = 0$ for all $j < 0$, then we say that z_0 is a *removable singularity* of $f(z)$.

(ii) If $a_{-m} \neq 0$ for some $m \in \mathbb{N}$, but $a_j = 0$ for all $j < -m$, then we say that z_0 is a *pole of order* m for $f(z)$.

(iii) If $a_j \neq 0$ for infinitely many $j < 0$, then we say that z_0 is an *essential singularity* of $f(z)$.

Example 29.2. Suppose that

$$f(z) = \frac{\sin z}{z}$$

for $|z| > 0$. Since the Laurent series expansion of $f(z)$ is

$$f(z) = \frac{1}{z}\sin z = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots,$$

we conclude that $z_0 = 0$ is a removable singularity.

Example 29.3. Suppose that

$$f(z) = \frac{e^z}{z^m}$$

for $|z| > 0$ where m is a positive integer. Since

$$\begin{aligned} f(z) &= \frac{1}{z^m}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) \\ &= \frac{1}{z^m} + \frac{1}{z^{m-1}} + \frac{1}{2!z^{m-2}} + \cdots + \frac{1}{(m-1)!z} + \frac{1}{m!} + \frac{z}{(m+1)!} + \cdots, \end{aligned}$$

we conclude that $z_0 = 0$ is a pole of order m.

Example 29.4. Suppose that

$$f(z) = e^{1/z}$$

for $|z| > 0$. Since

$$f(z) = e^{1/z} = 1 + (1/z) + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots,$$

we conclude that $z_0 = 0$ is an essential singularity.

Recall from Lecture #27 that we took a first look at residue theory as an application of Laurent series. The basic idea is that if we have the Laurent series expansion of a function $f(z)$ for $0 < |z - z_0| < R$ and C is an closed contour oriented counterclockwise surrounding z_0, then

$$\int_C f(z)\,\mathrm{d}z = 2\pi i a_{-1}$$

where a_{-1} is the coefficient of the $(z - z_0)^{-1}$ term in the Laurent series expansion of $f(z)$.

Definition. Suppose that the function $f(z)$ has an isolated singularity at z_0. The coefficient a_{-1} of $(z - z_0)^{-1}$ in the Laurent series expansion of $f(z)$ around z_0 is called the *residue of $f(z)$ at z_0* and is denoted by

$$a_{-1} = \operatorname{Res}(f; z_0).$$

Example 29.5. Suppose that $C = \{|z| = 1\}$ denotes the unit circle oriented counterclockwise. Compute the following three integrals:

(a) $\displaystyle\int_C \frac{\sin z}{z}\,\mathrm{d}z$,

(b) $\displaystyle\int_C \frac{e^z}{z^m}\,\mathrm{d}z$, where m is a positive integer, and

(c) $\displaystyle\int_C e^{1/z}\,\mathrm{d}z$.

Solution. In order to compute all three integrals, we use the fact that $z_0 = 0$ is an isolated singularity so that

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}(f; 0).$$

(a) From Example 29.2, we know that $a_{-1} = 0$ so that

$$\int_C \frac{\sin z}{z}\,\mathrm{d}z = 0.$$

(b) From Example 29.3, we know that $a_{-1} = 1/(m-1)!$ so that

$$\int_C \frac{e^z}{z^m}\,\mathrm{d}z = \frac{2\pi i}{(m-1)!}.$$

(c) From Example 29.4, we know that $a_{-1} = 1$ so that

$$\int_C e^{1/z}\,\mathrm{d}z = 2\pi i.$$

Note that although we could have used the Cauchy Integral Formula to solve **(a)** and **(b)**, we could not have used it to solve **(c)**.

Question. Given the obvious importance of the coefficient a_{-1} in a Laurent series, it is natural to ask if there is any way to determine a_{-1} without computing the entire Laurent series.

Theorem 29.6. *A function $f(z)$ has a pole of order m at z_0 if and only if*

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

for some function $g(z)$ that is analytic in a neighbourhood of z_0 and has $g(z_0) \neq 0$.

Proof. Suppose that $f(z)$ has a pole of order m at z_0. By definition, the Laurent series for $f(z)$ has the form

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \sum_{j=-(m-1)}^{\infty} a_j (z - z_0)^j$$

and so

$$(z - z_0)^m f(z) = a_{-m} + \sum_{j=-(m-1)}^{\infty} a_j (z - z_0)^{j+m} = a_{-m} + \sum_{j=1}^{\infty} a_{j-m} (z - z_0)^j.$$

Therefore, if we let

$$g(z) = a_{-m} + \sum_{j=1}^{\infty} a_{j-m} (z - z_0)^j,$$

then $g(z)$ is analytic in a neighbourhood of z_0. By assumption, $a_{-m} \neq 0$ since $f(z)$ has a pole of order m at z_0, and so $g(z_0) = a_{-m} \neq 0$.

Conversely, suppose that

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

for some function $g(z)$ that is analytic in a neighbourhood of z_0 and has $g(z_0) \neq 0$. Since $g(z)$ is analytic, it can be expanded in a Taylor series about z_0, say

$$g(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \cdots = \sum_{j=0}^{\infty} b_j (z - z_0)^j.$$

Since $g(z_0) = b_0 \neq 0$ by assumption, we obtain

$$f(z) = \frac{1}{(z - z_0)^m} \sum_{j=0}^{\infty} b_j (z - z_0)^j = \frac{b_0}{(z - z_0)^m} + \frac{b_1}{(z - z_0)^{m-1}} + \cdots .$$

Therefore, by definition, $f(z)$ has a pole of order m at z_0. ∎

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Prof. Michael Kozdron

Lecture #30: The Cauchy Residue Theorem

Recall that last class we showed that a function $f(z)$ has a pole of order m at z_0 if and only if

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

for some function $g(z)$ that is analytic in a neighbourhood of z_0 and has $g(z_0) \neq 0$.

Example 30.1. Suppose that

$$f(z) = \frac{\sin z}{(z^2 - 1)^2}.$$

Determine the order of the pole at $z_0 = 1$.

Solution. Observe that $z^2 - 1 = (z - 1)(z + 1)$ and so

$$f(z) = \frac{\sin z}{(z^2 - 1)^2} = \frac{\sin z}{(z - 1)^2 (z + 1)^2} = \frac{\sin z/(z + 1)^2}{(z - 1)^2}.$$

Since

$$g(z) = \frac{\sin z}{(z + 1)^2}$$

is analytic at 1 and $g(1) = 2^{-2} \sin(1) \neq 0$, we conclude that $z_0 = 1$ is a pole of order 2.

Suppose that $f(z)$ has a pole of order m at z_0 so that we can write

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-(m-1)}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0} + \sum_{j=0}^{\infty} a_j (z - z_0)^j.$$

Therefore,

$$(z - z_0)^m f(z) = a_{-m} + a_{-(m-1)}(z - z_0) + \cdots + a_{-1}(z - z_0)^{m-1} + \sum_{j=0}^{\infty} a_j (z - z_0)^{j+m}$$

which is analytic at z_0 by Theorem 29.6. This means we can pick out a_{-1} by taking successive derivatives. That is, differentiating $m - 1$ times gives

$$\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z - z_0)^m f(z) = (m - 1)! a_{-1} + \sum_{j=0}^{\infty} b_j (z - z_0)^{j+1}$$

for some coefficients b_j. The exact value of these coefficients is unimportant since we are going to make the non-constant terms disappear now. If we evaluate the previous expression at $z = z_0$ (which is justified since $(z - z_0)^m f(z) = g(z)$ is analytic at z_0), then we find

$$\left. \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z - z_0)^m f(z) \right|_{z=z_0} = (m - 1)! a_{-1},$$

and so we've found our formula for a_{-1} in the case that $f(z)$ has a pole of order m at z_0. Note that if $m = 1$, in which case $f(z)$ has a simple pole at z_0, then taking derivatives is unnecessary. We just need to evaluate $(z - z_0)f(z)$ at z_0.

Theorem 30.2. *If $f(z)$ is analytic for $0 < |z - z_0| < R$ and has a pole of order m at z_0, then*

$$\operatorname{Res}(f; z_0) = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z - z_0)^m f(z) \bigg|_{z=z_0} = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z - z_0)^m f(z).$$

In particular, if z_0 is a simple pole, then

$$\operatorname{Res}(f; z_0) = (z - z_0) f(z) \bigg|_{z=z_0} = \lim_{z \to z_0} (z - z_0) f(z).$$

Example 30.3. Determine the residue at $z_0 = 1$ of

$$f(z) = \frac{\sin z}{(z^2 - 1)^2}$$

and compute

$$\int_C f(z)\,\mathrm{d}z$$

where $C = \{|z - 1| = 1/2\}$ is the circle of radius $1/2$ centred at 1 oriented counterclockwise.

Solution. Since we can write $(z - 1)^2 f(z) = g(z)$ where

$$g(z) = \frac{\sin z}{(z + 1)^2}$$

is analytic at $z_0 = 1$ with $g(1) \neq 0$, the residue of $f(z)$ at $z_0 = 1$ is

$$\begin{aligned}
\operatorname{Res}(f; 1) = \frac{1}{(2-1)!} \frac{\mathrm{d}^{2-1}}{\mathrm{d}z^{2-1}} (z - 1)^2 f(z) \bigg|_{z=1} &= \frac{\mathrm{d}}{\mathrm{d}z} (z - 1)^2 f(z) \bigg|_{z=1} \\
&= \frac{\mathrm{d}}{\mathrm{d}z} \frac{\sin z}{(z + 1)^2} \bigg|_{z=1} \\
&= \frac{(z + 1)^2 \cos z - 2(z + 1) \sin z}{(z + 1)^4} \bigg|_{z=1} \\
&= \frac{4 \cos 1 - 4 \sin 1}{16} \\
&= \frac{\cos 1 - \sin 1}{4}.
\end{aligned}$$

Observe that if $C = \{|z - 1| = 1/2\}$ oriented counterclockwise, then the only singularity of $f(z)$ inside C is at $z_0 = 1$. Therefore,

$$\int_C \frac{\sin z}{(z^2 - 1)^2}\,\mathrm{d}z = 2\pi i \operatorname{Res}(f; 1) = \frac{(\cos 1 - \sin 1)\pi i}{2}.$$

It is worth pointing out that we could have also obtained this solution using the Cauchy Integral Formula; that is,

$$\int_C \frac{\sin z}{(z^2-1)^2}\,\mathrm{d}z = \int_C \frac{g(z)}{(z-1)^2}\,\mathrm{d}z = 2\pi i g'(1) = 2\pi i \cdot \frac{\cos 1 - \sin 1}{4} = \frac{(\cos 1 - \sin 1)\pi i}{2}$$

as above.

Remark. Suppose that C is a closed contour oriented counterclockwise. If $f(z)$ is analytic inside and on C except for a single point z_0 where $f(z)$ has a pole of order m, then both the Cauchy Integral Formula and the residue formula will require exactly the same work, namely the calculation of the $m-1$ derivative of $(z-z_0)^m f(z)$.

Recall that there are two other types of isolated singular points to consider, namely removable singularities and essential singularities. If the singularity is removable, then the residue is obviously 0. Unfortunately, there is no direct way to determine the residue associated with an essential singularity. The coefficient a_{-1} of the Laurent series must be determined explicitly.

In summary, suppose that $f(z)$ is analytic for $0 < |z-z_0| < R$ and has an isolated singularity at z_0. By direct inspection of the function, one may make an educated guess as to whether the isolated singularity is removable, a pole, or essential. If it believed to be either removable or essential, then compute the Laurent series to determine $\mathrm{Res}(f; z_0)$. If it is believed to be a pole, then attempt to compute $\mathrm{Res}(f; z_0)$ using Theorem 30.2.

Theorem 30.4 (Cauchy Residue Theorem). *Suppose that C is a closed contour oriented counterclockwise. If $f(z)$ is analytic inside and on C except at a finite number of isolated singularities $z_1, z_2, \ldots, z_n$, then*

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f; z_j).$$

Proof. Observe that if C is a closed contour oriented counterclockwise, then integration over C can be continuously deformed to a union of integrations over $C_1, C_2, \ldots, C_n$ where C_j is a circle oriented counterclockwise encircling exactly one isolated singularity, namely z_j, and not passing through any of the other isolated singular points. This yields

$$\int_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \cdots + \int_{C_n} f(z)\,\mathrm{d}z.$$

Since

$$\int_{C_j} f(z)\,\mathrm{d}z = 2\pi i \,\mathrm{Res}(f; z_j),$$

the proof is complete. □

Remark. Note that if the isolated singularities of $f(z)$ inside C are all either removable or poles, then the Cauchy Integral Formula can also be used to compute

$$\int_C f(z)\,\mathrm{d}z.$$

If any of the isolated singularities are essential, then the Cauchy Integral Formula does not apply. Moreover, even when $f(z)$ has only removable singularities or poles, the Residue Theorem is often much easier to use than the Cauchy Integral Formula.

Example 30.5. Compute

$$\int_C \frac{3z^3 + 4z^2 - 5z + 1}{(z - 2i)(z^3 - z)}\,\mathrm{d}z$$

where $C = \{|z| = 3\}$ is the circle of radius 3 centred at 0 oriented counterclockwise.

Solution. Observe that

$$f(z) = \frac{3z^3 + 4z^2 - 5z + 1}{(z - 2i)(z^3 - z)} = \frac{3z^3 + 4z^2 - 5z + 1}{z(z-1)(z+1)(z-2i)}$$

has isolated singular points at $z_1 = 0$, $z_2 = 1$, $z_3 = -1$, and $z_4 = 2i$. Moreover, each isolated singularity is a simple pole, and so

$$\mathrm{Res}(f; 0) = \left.\frac{3z^3 + 4z^2 - 5z + 1}{(z-1)(z+1)(z-2i)}\right|_{z=0} = \frac{1}{-1 \cdot -2i} = -\frac{i}{2},$$

$$\mathrm{Res}(f; 1) = \left.\frac{3z^3 + 4z^2 - 5z + 1}{z(z+1)(z-2i)}\right|_{z=1} = \frac{3 + 4 - 5 + 1}{1 \cdot 2 \cdot (1 - 2i)} = \frac{3}{2(1-2i)} = \frac{3(1+2i)}{10},$$

$$\mathrm{Res}(f; -1) = \left.\frac{3z^3 + 4z^2 - 5z + 1}{z(z-1)(z-2i)}\right|_{z=-1} = \frac{-3 + 4 + 5 + 1}{-1 \cdot -2 \cdot (-1 - 2i)} = -\frac{7}{2(1+2i)} = \frac{7(2i-1)}{10},$$

$$\mathrm{Res}(f; 2i) = \left.\frac{3z^3 + 4z^2 - 5z + 1}{z(z-1)(z+1)}\right|_{z=2i} = \frac{3(2i)^3 + 4(2i)^2 - 5(2i) + 1}{2i(2i-1)(2i+1)} = \frac{34 - 15i}{10}.$$

By the Cauchy Residue Theorem,

$$\int_C \frac{3z^3 + 4z^2 - 5z + 1}{(z - 2i)(z^3 - z)}\,\mathrm{d}z = 2\pi i \left(-\frac{i}{2} + \frac{3(1+2i)}{10} + \frac{7(2i-1)}{10} + \frac{34 - 15i}{10} \right) = 6\pi i.$$

Lecture #31: Computing Real Trigonometric Integrals

Suppose that C is a closed contour oriented counterclockwise. Last class we proved the Residue Theorem which states that if $f(z)$ is analytic inside and on C, except for a finite number of isolated singular points $z_1, \dots, z_n$, then

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f; z_j).$$

In order to compute $\mathrm{Res}(f; z_j)$, we need to determine whether or not the isolated singular point z_j is removable, essential, or a pole of order m. If z_j is a pole of order m, then we know

$$\mathrm{Res}(f; z_j) = \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}(z-z_j)^m f(z)\bigg|_{z=z_j} = \frac{1}{(m-1)!}\lim_{z\to z_j}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}(z-z_j)^m f(z).$$

In particular, if z_j is a simple pole, then

$$\mathrm{Res}(f; z_j) = (z-z_j)f(z)\bigg|_{z=z_j} = \lim_{z\to z_j}(z-z_j)f(z).$$

Example 31.1. Compute

$$\int_C \frac{2}{z^2+4z+1}\,\mathrm{d}z$$

where $C = \{|z| = 1\}$ is the circle of radius 1 centred at 0 oriented counterclockwise.

Solution. Clearly

$$f(z) = \frac{2}{z^2+4z+1}$$

has two simple poles. Notice that $z^2+4z+1 = (z^2+4z+4)-3 = (z+2)^2-3 = 0$ implies

$$z_1 = \sqrt{3}-2 \quad \text{and} \quad z_2 = -\sqrt{3}-2$$

are simple poles. However, $|z_1| < 1$ and $|z_2| > 1$ which means that $f(z)$ only has one isolated singularity inside C. Since

$$z^2+4z+1 = (z-z_1)(z-z_2) = (z-\sqrt{3}+2)(z+\sqrt{3}+2),$$

we find

$$\mathrm{Res}(f; z_1) = \mathrm{Res}(f; \sqrt{3}-2) = \frac{2}{z+\sqrt{3}+2}\bigg|_{z=\sqrt{3}-2} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}},$$

so by the Cauchy Residue Theorem,

$$\int_C \frac{2z}{z^2+4z+1}\,\mathrm{d}z = 2\pi i \cdot \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}\pi i.$$

Suppose that we now parametrize C by $z(t) = e^{it}$, $0 \le t \le 2\pi$, and attempt to compute this same contour integral using

$$\int_C f(z)\,\mathrm{d}z = \int_0^{2\pi} f(e^{it}) \cdot ie^{it}\,\mathrm{d}t.$$

That is,

$$\begin{aligned}
\int_C \frac{2}{z^2+4z+1}\,\mathrm{d}z = \int_0^{2\pi} \frac{2}{e^{2it}+4e^{it}+1} \cdot ie^{it}\,\mathrm{d}t &= 2i \int_0^{2\pi} \frac{e^{it}}{e^{2it}+4e^{it}+1}\,\mathrm{d}t \\
&= 2i \int_0^{2\pi} \frac{1}{e^{it}+4+e^{-it}}\,\mathrm{d}t \\
&= 2i \int_0^{2\pi} \frac{1}{4+2\cos t}\,\mathrm{d}t \\
&= i \int_0^{2\pi} \frac{1}{2+\cos t}\,\mathrm{d}t
\end{aligned}$$

and so

$$\frac{2}{\sqrt{3}}\pi i = i \int_0^{2\pi} \frac{1}{2+\cos t}\,\mathrm{d}t \quad \text{or, equivalently,} \quad \int_0^{2\pi} \frac{1}{2+\cos t}\,\mathrm{d}t = \frac{2}{\sqrt{3}}\pi.$$

Notice that we were able to compute a definite integral by relating it to a contour integral that could be evaluated using the Residue Theorem. If we have a definite integral, the limits of integration are 0 to 2π, and the integrand is a function of $\cos\theta$ and $\sin\theta$, then we can systematically convert it to a contour integral as follows. Let $C = \{|z| = 1\}$ denote the circle of radius 1 centred at 0 oriented counterclockwise and parametrize C by $z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$, so that $z'(\theta) = ie^{i\theta} = iz(\theta)$. Since

$$z(\theta) = e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad \frac{1}{z(\theta)} = e^{-i\theta} = \cos\theta - i\sin\theta,$$

we find

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z(\theta) + \frac{1}{z(\theta)}\right) \quad \text{and} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z(\theta) - \frac{1}{z(\theta)}\right).$$

Therefore,

$$\begin{aligned}
\int_0^{2\pi} F(\cos\theta, \sin\theta)\,\mathrm{d}\theta &= \int_0^{2\pi} F\left(\frac{1}{2}\left(z(\theta)+\frac{1}{z(\theta)}\right), \frac{1}{2i}\left(z(\theta)-\frac{1}{z(\theta)}\right)\right)\mathrm{d}\theta \\
&= \int_0^{2\pi} F\left(\frac{1}{2}\left(z(\theta)+\frac{1}{z(\theta)}\right), \frac{1}{2i}\left(z(\theta)-\frac{1}{z(\theta)}\right)\right)\frac{z'(\theta)}{z'(\theta)}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} F\left(\frac{1}{2}\left(z(\theta)+\frac{1}{z(\theta)}\right), \frac{1}{2i}\left(z(\theta)-\frac{1}{z(\theta)}\right)\right)\frac{1}{iz(\theta)}z'(\theta)\,\mathrm{d}\theta \\
&= \int_C F\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{1}{iz}\,\mathrm{d}z.
\end{aligned}$$

Example 31.2. Compute
$$\int_0^{2\pi} \frac{\cos(2\theta)}{5 - 4\cos\theta}\,\mathrm{d}\theta.$$

Solution. Let $C = \{|z| = 1\}$ oriented counterclockwise be parametrized by $z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$. Note that
$$\cos(2\theta) = \frac{e^{2i\theta} + e^{-2i\theta}}{2} = \frac{1}{2}\left(z(\theta)^2 + \frac{1}{z(\theta)^2}\right) = \frac{z(\theta)^4 + 1}{2z(\theta)^2}$$
and
$$5 - 4\cos\theta = 5 - 4 \cdot \frac{1}{2}\left(z(\theta) + \frac{1}{z(\theta)}\right) = 5 - 2z(\theta) - \frac{2}{z(\theta)} = -\frac{2z(\theta)^2 - 5z(\theta) + 2}{z(\theta)},$$
so that
$$\frac{\cos(2\theta)}{5 - 4\cos\theta} = \frac{\frac{z(\theta)^4+1}{2z(\theta)^2}}{-\frac{2z(\theta)^2-5z(\theta)+2}{z(\theta)}} = -\frac{z(\theta)^4 + 1}{2z(\theta)\left(2z(\theta)^2 - 5z(\theta) + 2\right)} = -\frac{z(\theta)^4 + 1}{2z(\theta)(2z(\theta) - 1)(z(\theta) - 2)}.$$
This implies
$$\begin{aligned}\int_0^{2\pi} \frac{\cos(2\theta)}{5 - 4\cos\theta}\,\mathrm{d}\theta = -\int_0^{2\pi} \frac{z(\theta)^4 + 1}{2z(\theta)(2z(\theta) - 1)(z(\theta) - 2)}\,\mathrm{d}\theta &= -\int_C \frac{z^4 + 1}{2z(2z - 1)(z - 2)} \cdot \frac{1}{iz}\,\mathrm{d}z \\ &= \frac{i}{2}\int_C \frac{z^4 + 1}{z^2(2z - 1)(z - 2)}\,\mathrm{d}z.\end{aligned}$$
Let
$$f(z) = \frac{z^4 + 1}{z^2(2z - 1)(z - 2)}$$
so that $f(z)$ clearly has a double pole at $z_1 = 0$, a simple pole at $z_2 = 1/2$, and a simple pole at $z_3 = 2$. Of these three singularities of $f(z)$, only two of them are inside C. Therefore,
$$\operatorname{Res}(f; 0) = \frac{\mathrm{d}}{\mathrm{d}z}\frac{z^4 + 1}{(2z - 1)(z - 2)}\bigg|_{z=0} = \frac{4z^3(2z - 1)(z - 2) - (z^4 + 1)(4z - 5)}{(2z - 1)^2(z - 2)^2}\bigg|_{z=0} = \frac{5}{4}$$
and
$$\operatorname{Res}(f; 1/2) = \left(z - \frac{1}{2}\right)\frac{z^4 + 1}{z^2(2z - 1)(z - 2)}\bigg|_{z=1/2} = \frac{z^4 + 1}{2z^2(z - 2)}\bigg|_{z=1/2} = -\frac{17}{12}.$$
By the Residue Theorem,
$$\int_0^{2\pi} \frac{\cos(2\theta)}{5 - 4\cos\theta}\,\mathrm{d}\theta = \frac{i}{2}\int_C \frac{z^4 + 1}{z^2(2z - 1)(z - 2)}\,\mathrm{d}z = \frac{i}{2} \cdot 2\pi i\left(\frac{5}{4} - \frac{17}{12}\right) = \frac{\pi}{6}.$$
Remark. It is possible to compute both
$$\int_0^{2\pi} \frac{1}{2 + \cos\theta}\,\mathrm{d}\theta \quad \text{and} \quad \int_0^{2\pi} \frac{\cos(2\theta)}{5 - 4\cos\theta}\,\mathrm{d}\theta$$
by calculating indefinite Riemann integrals. However, such calculations are *very* challenging. The contour integral approach is significantly easier.

Lecture #32: Computing Real Improper Integrals

Example 32.1. Compute

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x.$$

Solution. Although it is possible to compute this particular integral using partial fractions easily enough, we will solve it with complex variables in order to illustrate a general method which works in more complicated cases. Observe that by symmetry,

$$2\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x = \int_{-\infty}^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x.$$

Suppose that $C = C_R \oplus [-R, R]$ denotes the closed contour oriented counterclockwise obtained by concatenating C_R, that part of the circle of radius R in the upper half plane parametrized by $z(t) = Re^{it}$, $0 \le t \le \pi$, with $[-R, R]$, the line segment along the real axis connecting the point $-R$ to the point R. Therefore, if

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)},$$

then

$$\int_C f(z)\,\mathrm{d}z = \int_{[-R,R]} f(z)\,\mathrm{d}z + \int_{C_R} f(z)\,\mathrm{d}z. \tag{$*$}$$

We now observe that we can compute

$$\int_C f(z)\,\mathrm{d}z = \int_C \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z$$

using the Residue Theorem. That is, since

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)} = \frac{z^2}{(z+i)(z-i)(z+2i)(z-2i)},$$

we find that $f(z)$ has simple poles at $z_1 = i$, $z_2 = -i$, $z_3 = 2i$, $z_4 = -2i$. However, only z_1 and z_3 are inside C (assuming, of course, that R is sufficiently large). Thus,

$$\mathrm{Res}(f; z_1) = \left.\frac{z^2}{(z+i)(z+2i)(z-2i)}\right|_{z=z_1=i} = \frac{i^2}{(2i)(3i)(-i)} = \frac{i}{6}$$

and

$$\mathrm{Res}(f; z_3) = \left.\frac{z^2}{(z+i)(z-i)(z+2i)}\right|_{z=z_3=2i} = \frac{(2i)^2}{(3i)(i)(4i)} = -\frac{i}{3},$$

so by the Residue Theorem,

$$\int_C f(z)\,\mathrm{d}z = \int_C \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z = 2\pi i\left(\frac{i}{6} - \frac{i}{3}\right) = \frac{\pi}{3}.$$

In other words, we have shown that for R sufficiently large $(*)$ becomes

$$\frac{\pi}{3} = \int_{[-R,R]} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z + \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z.$$

The next step is to observe that since $[-R,R]$ denotes the line segment along the real axis connecting the point $-R$ to the point R, if we parametrize the line segment by $z(t)=t$, $-R \le t \le R$, then since $z'(t)=1$, we obtain

$$\int_{[-R,R]} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z = \int_{-R}^{R} \frac{t^2}{(t^2+1)(t^2+4)}\,\mathrm{d}t = \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x$$

where the last equality follows by a simple change of dummy variable from t to x. Thus,

$$\frac{\pi}{3} = \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x + \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z$$

and so by taking the limit as $R\to\infty$ of both sides we obtain

$$\frac{\pi}{3} = \lim_{R\to\infty}\int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x + \lim_{R\to\infty}\int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z.$$

We now make two claims.

Claim 1.

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x$$

Claim 2.

$$\lim_{R\to\infty}\int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z = 0$$

Assuming that both claims are true, we obtain

$$\frac{\pi}{3} = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x \quad \text{and so} \quad \int_{0}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x = \frac{\pi}{6}$$

which is in agreement with what one obtains by using partial fractions.

Hence, the next task is to address these two claims. We will begin with the second claim which follows immediately from this result.

Theorem 32.2. *Suppose that C_R denotes the upper half of the circle connecting R to $-R$ and parametrized by $z(t)=Re^{it}$, $0\le t\le\pi$. If*

$$f(z) = \frac{P(z)}{Q(z)}$$

is the ratio of two polynomials satisfying $\deg Q \ge \deg P + 2$, then

$$\lim_{R\to\infty}\int_{C_R} f(z) = 0.$$

Proof. The fact that $\deg Q \geq \deg P + 2$ implies that if $|z|$ is sufficiently large, then

$$|f(z)| = \left|\frac{P(z)}{Q(z)}\right| \leq \frac{K}{|z|^2} \tag{$\dagger$}$$

for some constant $K < \infty$. (See the supplement for derivation of (†).) Hence,

$$\left|\int_{C_R} f(z)\,\mathrm{d}z\right| \leq \int_{C_R} |f(z)|\,\mathrm{d}z = \int_{C_R} \frac{K}{|z|^2}\,\mathrm{d}z = \frac{K}{R^2}\ell(C_R) = \frac{K\pi}{R}$$

since $\ell(C_R) = \pi R$ is the arclength of C_R. Taking $R \to \infty$ then yields the result. □

Thus, we conclude from this theorem that

$$\lim_{R\to\infty} \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z = 0$$

since $P(z) = z^2$ has degree 2 and $Q(z) = (z^2+1)(z^2+4)$ has degree 4.

Review of Improper Integrals

In order to discuss Claim 1, it is necessary to review improper integrals from first year calculus. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. By definition,

$$\int_0^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty} \int_0^b f(x)\,\mathrm{d}x.$$

Thus, assuming the limit exists as a real number, we define

$$\int_0^\infty f(x)\,\mathrm{d}x$$

to be this limiting value. By definition,

$$\int_{-\infty}^0 f(x)\,\mathrm{d}x = \lim_{a\to-\infty} \int_a^0 f(x)\,\mathrm{d}x.$$

Thus, assuming the limit exists as a real number, we define

$$\int_{-\infty}^0 f(x)\,\mathrm{d}x$$

to be this limiting value. By definition,

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_0^\infty f(x)\,\mathrm{d}x + \int_{-\infty}^0 f(x)\,\mathrm{d}x = \lim_{b\to\infty} \int_0^b f(x)\,\mathrm{d}x + \lim_{a\to-\infty} \int_a^0 f(x)\,\mathrm{d}x.$$

Thus, in order for

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x \tag{$*$}$$

to exist it must be the case that both

$$\lim_{b\to\infty}\int_0^b f(x)\,\mathrm{d}x \quad \text{and} \quad \lim_{a\to-\infty}\int_a^0 f(x)\,\mathrm{d}x$$

exist as real numbers. However, if one of these limits fails to exist as a real number, then the improper integral ($*$) does not exist. Sometimes, we might write

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{a\to-\infty,b\to\infty}\int_a^b f(x)\,\mathrm{d}x \tag{$\dagger$}$$

instead which just writes the two separate limits in a single piece of notation. It is important to stress that this notation still implies that two separate limits are being taken: $a \to -\infty$ and $b \to \infty$. It might be tempting to try and combine the two separate limits into a single limit as follows:

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{c\to\infty}\int_{-c}^{c} f(x)\,\mathrm{d}x. \tag{$\ddagger$}$$

However, ($\dagger$) and ($\ddagger$) are not the same! As we will now show, it is possible for

$$\lim_{c\to\infty}\int_{-c}^{c} f(x)\,\mathrm{d}x$$

to exist, but for

$$\lim_{a\to-\infty,b\to\infty}\int_a^b f(x)\,\mathrm{d}x$$

not to exist.

Example 32.3. Observe that

$$\int_{-c}^{c} x\,\mathrm{d}x = \left.\frac{x^2}{2}\right|_{-c}^{c} = \frac{c^2}{2} - \frac{(-c)^2}{2} = 0$$

and so

$$\lim_{c\to\infty}\int_{-c}^{c} x\,\mathrm{d}x = \lim_{c\to 0} 0 = 0.$$

On the other hand,

$$\int_a^0 x\,\mathrm{d}x = \left.\frac{x^2}{2}\right|_a^0 = -\frac{a^2}{2} \quad \text{and} \quad \int_0^b x\,\mathrm{d}x = \left.\frac{x^2}{2}\right|_0^b = \frac{b^2}{2}$$

so that

$$\lim_{a\to-\infty}\int_a^0 x\,\mathrm{d}x = -\lim_{a\to-\infty}\frac{a^2}{2} = -\infty \quad \text{and} \quad \lim_{b\to\infty}\int_0^b x\,\mathrm{d}x = \lim_{b\to\infty}\frac{b^2}{2} = \infty.$$

Thus,

$$\lim_{a\to-\infty,b\to\infty}\int_a^b x\,\mathrm{d}x = -\infty + \infty = \infty - \infty$$

so that

$$\int_{-\infty}^{\infty} x\,\mathrm{d}x$$

does not exist.

Supplement: Verification of (†) from proof of Theorem 32.2

Suppose that $Q(z) = b_0 + b_1 z + \cdots + b_m z^m$ is a polynomial of degree m. Without loss of generality assume that $b_m = 1$. Therefore,

$$z^{-m}Q(z) = 1 + \frac{b_{m-1}}{z} + \cdots + \frac{b_0}{z^m}$$

and so by the triangle inequality,

$$|z^{-m}||Q(z)| = \left|1 + \frac{b_{m-1}}{z} + \cdots + \frac{b_0}{z^m}\right| \geq 1 - \left|\frac{b_{m-1}}{z} + \cdots + \frac{b_0}{z^m}\right|.$$

Let $M = \max\{1, |b_0|, \ldots, |b_{m-1}|\}$ and note that $2mM > 1$. This means that if $|z| \geq 2mM$, then

$$\left|\frac{b_{m-j}}{z^j}\right| \leq \frac{M}{|z|^j} \leq \frac{M}{|z|} \leq \frac{1}{2m}.$$

Therefore, since there are m terms in the following sum,

$$\left|\frac{b_{m-1}}{z} + \cdots + \frac{b_0}{z^m}\right| \leq \frac{1}{2m} + \frac{1}{2m} + \cdots + \frac{1}{2m} = \frac{1}{2}$$

which implies that

$$|z^{-m}||Q(z)| \geq 1 - \left|\frac{b_{m-1}}{z} + \cdots + \frac{b_0}{z^m}\right| \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

Hence, we obtain,

$$|Q(z)| \geq \frac{|z|^m}{2}$$

for $|z|$ sufficiently large. On the other hand, suppose that $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial of degree n so that

$$z^{-n}P(z) = a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}.$$

If $|z| > 1$, then by the triangle inequality,

$$|z^{-n}P(z)| \leq |a_n| + \left|\frac{a_{n-1}}{z}\right| + \cdots + \left|\frac{a_0}{z^n}\right| \leq |a_n| + |a_{n-1}| + \cdots + |a_0|$$

and so with $C = |a_0| + \cdots + |a_n|$ we obtain

$$|z^{-n}P(z)| \leq C$$

for $|z| > 1$. Now suppose that

$$f(z) = \frac{P(z)}{Q(z)}$$

is the ratio of polynomials with $\deg Q(z) \geq \deg P(z) + 2$. If $\deg P(z) = n$ and $\deg Q(z) = n+k$ with $k \geq 2$, then we find that for $|z|$ sufficiently large,

$$\left|\frac{P(z)}{Q(z)}\right| = \frac{|z^{-n}P(z)|}{|z^{-n}Q(z)|} \leq \frac{C}{|z|^{-n}\frac{|z|^{n+k}}{2}} = \frac{2C}{|z|^k} \leq \frac{2C}{|z|^2} = \frac{K}{|z|^2}$$

since $k \geq 2$.

Lecture #33: Cauchy Principal Value

Definition. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function on $(-\infty, \infty)$. If

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x$$

exists, then we define the *Cauchy principal value of the integral of f over* $(-\infty,\infty)$ to be this value, and we write

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x$$

for the value of this limit.

Remark. If

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

exists, then

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \text{p.v.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

However,

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

may exist, even though

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

does not exist. For instance,

$$\text{p.v.} \int_{-\infty}^{\infty} x\,\mathrm{d}x = 0 \quad \text{whereas} \quad \int_{-\infty}^{\infty} x\,\mathrm{d}x \ \text{ does not exist.}$$

We can now finish verifying Claim 2 from Example 32.1 of the previous lecture.

Example 32.1 (continued). Recall that we had deduced

$$\frac{\pi}{3} = \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x + \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z$$

where C_R is that part of the circle of radius R in the upper half plane parametrized by $z(t) = Re^{it}$, $0 \le t \le \pi$. Taking the limit as $R \to \infty$ and using Theorem 32.2, we obtained

$$\frac{\pi}{3} = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x.$$

By the definition of the Cauchy principal value, we have actually shown

$$\frac{\pi}{3} = \text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x.$$

We now observe that

$$\frac{x^2}{(x^2+1)(x^2+4)}$$

is an even function so that

$$\int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x = 2\int_{0}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x$$

which implies that

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x = 2\lim_{R\to\infty}\int_{0}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x.$$

In order to verify that the improper integral actually exists, note that

$$\left|\int_{0}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x\right| = \int_{0}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x \le \int_{0}^{R} \frac{1}{x^2+1}\,\mathrm{d}x = \arctan R$$

using the inequality $x^2 \le (x^2+4)$. Since $\arctan R \to \pi/2$ as $R \to \infty$, we conclude

$$\lim_{R\to\infty}\int_{0}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x$$

exists by the integral comparison test. Thus,

$$\frac{\pi}{3} = \text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x = 2\int_{0}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x$$

so that

$$\int_{0}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x = \frac{\pi}{6}.$$

Example 33.1. Compute

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{1}{x^2+2x+1}\,\mathrm{d}x.$$

Solution. Suppose that $C = C_R \oplus [-R, R]$ denotes the closed contour oriented counter-clockwise obtained by concatenating C_R, that part of the circle of radius R in the upper half plane parametrized by $z(t) = Re^{it}$, $0 \le t \le \pi$, with $[-R, R]$, the line segment along the real axis connecting the point $-R$ to the point R. Suppose further that

$$f(z) = \frac{1}{z^2+2z+2}$$

so that $f(z)$ has two simple poles. These occur where

$$z^2 + 2z + 2 = z^2 + 2z + 1 + 1 = (z+1)^2 + 1 = 0,$$

namely at $z_1 = \sqrt{-1} - 1 = i - 1$ and $z_2 = -\sqrt{-1} - 1 = -i - 1 = -(i+1)$. Note that only z_1 is inside C, at least for R sufficiently large. Therefore, since

$$f(z) = \frac{1}{z^2 + 2z + 2} = \frac{1}{(z - z_1)(z - z_2)},$$

we conclude that

$$\operatorname{Res}(f; z_1) = \left.\frac{1}{z - z_2}\right|_{z = z_1} = \frac{1}{z_1 - z_2} = \frac{1}{i - i + (i+1)} = \frac{1}{2i}.$$

This implies

$$\int_C \frac{1}{z^2 + 2z + 2}\,\mathrm{d}z = 2\pi i \frac{1}{2i} = \pi$$

so that

$$\begin{aligned}\pi = \int_C \frac{1}{z^2 + 2z + 2}\,\mathrm{d}z &= \int_{[-R,R]} \frac{1}{z^2 + 2z + 2}\,\mathrm{d}z + \int_{C_R} \frac{1}{z^2 + 2z + 2}\,\mathrm{d}z \\ &= \int_{-R}^{R} \frac{1}{x^2 + 2x + 1}\,\mathrm{d}x + \int_{C_R} \frac{1}{z^2 + 2z + 2}\,\mathrm{d}z.\end{aligned}$$

Taking $R \to \infty$ yields

$$\pi = \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{x^2 + 2x + 1}\,\mathrm{d}x + \lim_{R \to \infty} \int_{C_R} \frac{1}{z^2 + 2z + 2}\,\mathrm{d}z = \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 1}\,\mathrm{d}x$$

using Theorem 32.2 to conclude that the second limit is 0.

Lecture #34: The Fundamental Theorem of Algebra

Theorem 34.1 (Fundamental Theorem of Algebra). *Every nonconstant polynomial with complex coefficients has at least one root.*

Proof. Suppose to the contrary that

$$P(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + a_m z^m$$

is a nonconstant polynomial of degree m (so that $m \geq 1$) having no roots. Without loss of generality, assume that $a_m = 1$. Consequently,

$$Q(z) = \frac{1}{P(z)}$$

must be an entire function. Let $M = \max\{1, |a_0|, |a_1|, \ldots, |a_{m-1}|\}$. Using an argument similar to that given in the supplement to the proof of Theorem 32.2, we find that if $|z| \geq 2mM$, then

$$|Q(z)| = \left|\frac{1}{P(z)}\right| \leq \frac{1}{|z|^m/2} \leq \frac{2}{(2mM)^m}.$$

On the other hand, if $|z| \leq 2mM$, then we have a continuous real-valued function, namely $|Q(z)|$, on a closed disk. From calculus, we conclude that the function must be bounded. Hence,

$$Q(z) = \frac{1}{P(z)}$$

is bounded and entire. Thus, $Q(z)$ must be constant and therefore $P(z)$ must itself be constant. However, we assumed that $m \geq 1$ so this contradicts our assumption and we conclude that $P(z)$ must have at least one root. $\square$

Corollary 34.2. *If $P(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + a_m z^m$ is a polynomial of degree m, then $P(z)$ has m complex roots. Moreover, if the coefficients $a_j \in \mathbb{R}$ for all j, then the roots come in complex conjugate pairs.*

Proof. By Theorem 34.1 we know that $P(z)$ has at least one root. Thus, suppose that z_1 is a root of $P(z)$. This means that we can write

$$P(z) = (z - z_1)(b_0 + b_1 z + \cdots + b_{m-1} z^{m-1})$$

for some coefficients $b_0, b_1, \ldots, b_{m-1}$. However, $Q(z) = b_0 + b_1 z + \cdots + b_{m-1} z^{m-1}$ is a polynomial of degree $m - 1$, and so applying Theorem 34.1 to this polynomial, we conclude that $Q(z)$ has at least one root. This means that $P(z)$ must have at least two roots. Continuing in this fashion yields m complex roots for $P(z)$.

Suppose that $P(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + a_m z^m$ with $a_j \in \mathbb{R}$ for all j. Suppose that z_0 is a root of $P(z)$ so that $P(z_0) = 0$. Consider $\overline{z_0}$. We find

$$\begin{aligned} P(\overline{z_0}) = a_0 + a_1 \overline{z_0} + \cdots + a_{m-1} (\overline{z_0})^{m-1} + a_m (\overline{z_0})^m &= a_0 + \overline{a_1 z_0} + \cdots + \overline{a_{m-1} z_0^{m-1}} + \overline{a_m z_0^m} \\ &= \overline{a_0 + a_1 z_0 + \cdots + a_{m-1} z_0^{m-1} + a_m z_0^m} \\ &= \overline{P(z_0)} \\ &= \overline{0} \\ &= 0. \end{aligned}$$

Thus, if z_0 is a root of $P(z)$, then so too is $\overline{z_0}$. In other words, the roots of a polynomial with real coefficients come in conjugate pairs. □

Example 34.3. Show that the roots of $P(z) = z^3 - 1$ come in conjugate pairs.

Solution. From Theorem 34.1 we know that $P(z)$ has three roots. They are $z_1 = 1$, $z_2 = e^{2\pi i/3}$, and $z_3 = e^{4\pi i/3}$. Observe that z_1 is real so that it is self-conjugate. Furthermore, $\overline{z_2} = e^{-2\pi i/3} = e^{4\pi i/3} = z_3$ so that z_2 and z_3 are a conjugate pair of roots.