Math 312 Fall 2012 Final Exam – Solutions

1. Since $e^z \neq 0$ for all $z \in \mathbb{C}$, we can multiply $e^z + 2e^{-z} = 3$ by e^z and simplify obtain $e^{2z} - 3e^z + 2 = 0$. Notice that $e^{2z} - 3e^z + 2 = (e^z - 2)(e^z - 1)$ and so $e^{2z} - 3e^z + 2 = 0$ iff either $e^z - 2 = 0$ or $e^z - 1 = 0$. Consider first the equation $e^z = 1$. Since $e^{2\pi k i} = 1$ for any $k \in \mathbb{Z}$, we conclude that $e^z - 1 = 0$ iff $z \in \{2\pi k i,\ k \in \mathbb{Z}\}$. Now consider $e^z = 2$. Since $e^{\log 2 + 2\pi k i} = 2$ for any $k \in \mathbb{Z}$, we conclude that $e^z - 2 = 0$ iff $z \in \{\log 2 + 2\pi k i,\ k \in \mathbb{Z}\}$. This implies that if $z \in \{2\pi k i,\ k \in \mathbb{Z}\} \cup \{\log 2 + 2\pi k i,\ k \in \mathbb{Z}\} = \{2\pi k i,\ \log 2 + 2\pi k i,\ k \in \mathbb{Z}\}$, then $e^z + 2e^{-z} = 3$. Since we are only interested in those z with $|z| < 10$, we see that

$$z \in \{0,\ 2\pi i,\ -2\pi i,\ \log 2,\ \log 2 + 2\pi i,\ \log 2 - 2\pi i\}.$$

2. Consider the function $g(z) = -iz$. Since the action of $g(z)$ is rotation clockwise by an angle of $\pi/2$, we see that the image of D under $g(z)$ is $E = \{z : \mathrm{Re}(z) < 0,\ 0 < \mathrm{Im}(z) < \pi/2\}$. Now let $f(z) = e^z$ so that $w = f(g(z))$. The image of D under w is exactly the image of E under $f(z)$. Observe that we can express E as $E = \{z = x + iy : x < 0 \text{ and } 0 < y < \pi/2\}$. Since $e^z = e^x e^{iy}$ and $x < 0$, we conclude that $|e^z| = e^x < 1$. Moreover, e^{iy} for $0 < y < \pi/2$ describes that part of the unit circle centred at 0 in the first quadrant. Thus, the image of D in the w-plane is exactly $\{w \in \mathbb{C} : |w| < 1,\ \ 0 < \mathrm{Arg}(w) < \pi/2\}$.

3. (i) We begin by observing that

$$f(z) = \frac{z}{z^2 - 4z + 3} = \frac{3/2}{z-3} - \frac{1/2}{z-1} = \frac{1/2}{1-z} - \frac{3/2}{3-z} = \frac{1/2}{1-z} - \frac{1/2}{1-z/3}.$$

Since

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j \quad \text{for } |z| < 1 \text{ and } \quad \frac{1}{1-z/3} = \sum_{j=0}^{\infty} 3^{-j} z^j \quad \text{for } |z| < 3,$$

we conclude that if $|z| < 1$, then

$$f(z) = \frac{1}{2}\sum_{j=0}^{\infty} z^j - \frac{1}{2}\sum_{j=0}^{\infty} 3^{-j} z^j = \sum_{j=0}^{\infty} \frac{1 - 3^{-j}}{2} z^j.$$

(ii) We now observe that

$$f(z) = \frac{z}{z^2 - 4z + 3} = \frac{3/2}{z-3} - \frac{1/2}{z-1} = -\frac{1/2}{1-z/3} - \frac{1}{2z}\frac{1}{1-1/z}.$$

Since

$$\frac{1}{2z}\frac{1}{1-1/z} = \frac{1}{2z}\sum_{j=0}^{\infty} z^{-j} = \frac{1}{2}\sum_{j=0}^{\infty} z^{-1-j} \quad \text{for } |z| > 1 \text{ and } \quad \frac{1}{1-z/3} = \sum_{j=0}^{\infty} 3^{-j} z^j \quad \text{for } |z| < 3,$$

we conclude that if $1 < |z| < 3$, then

$$f(z) = -\frac{1}{2}\sum_{j=0}^{\infty} 3^{-j} z^j - \frac{1}{2}\sum_{j=0}^{\infty} z^{-1-j} = -\frac{1}{2}\sum_{j=0}^{\infty} 3^{-j} z^j - \frac{1}{2}\sum_{j=1}^{\infty} z^{-j}.$$

(iii) We now observe that

$$f(z) = \frac{z}{z^2 - 4z + 3} = \frac{3/2}{z-3} - \frac{1/2}{z-1} = \frac{3}{2z}\frac{1}{1-3/z} - \frac{1}{2z}\frac{1}{1-1/z}.$$

Since

$$\frac{1}{2z}\frac{1}{1-1/z} = \frac{1}{2}\sum_{j=0}^{\infty} z^{-1-j}$$

for $|z| > 1$ and

$$\frac{3}{2z}\frac{1}{1-3/z} = \frac{3}{2z}\sum_{j=0}^{\infty} 3^j z^{-j} = \frac{1}{2}\sum_{j=0}^{\infty} 3^{j+1} z^{-1-j}$$

for $|z| > 3$, we conclude that if $|z| > 3$, then

$$f(z) = \frac{1}{2}\sum_{j=0}^{\infty} 3^{j+1} z^{-1-j} - \frac{1}{2}\sum_{j=0}^{\infty} z^{-1-j} = \frac{1}{2}\sum_{j=0}^{\infty} (3^{j+1} - 1) z^{-1-j} = \frac{1}{2}\sum_{j=1}^{\infty} \frac{3^j - 1}{z^j}.$$

4. Observe that $(z^2+1)^3 = (z-i)^3(z+i)^3$ so that

$$f(z) = \frac{1}{(z^2+1)^3} = \frac{1}{(z-i)^3(z+i)^3}$$

has poles of order 3 at $z_1 = i$ and $z_2 = -i$. Note that only z_1 is inside C. Since

$$\text{Res}(f(z); i) = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\frac{1}{(z+i)^3}\bigg|_{z=i} = \frac{1}{2}\frac{12}{(z+i)^5}\bigg|_{z=i} = \frac{6}{(2i)^5} = \frac{6}{32i} = \frac{3}{16i},$$

we conclude from the residue theorem that

$$\int_C \frac{1}{(z^2+1)^3}\,\mathrm{d}z = 2\pi i \frac{3}{16i} = \frac{3\pi}{8}.$$

5. Suppose that $C = \{|z| = 1\}$ oriented counterclockwise is parametrized by $z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$. Since $z'(\theta) = ie^{i\theta} = iz(\theta)$, we obtain

$$\int_0^{2\pi} \frac{\cos\theta}{5 - 4\cos\theta}\,\mathrm{d}\theta = \int_0^{2\pi} \frac{\frac{1}{2}(z(\theta) + z(\theta)^{-1})}{5 - 4\frac{1}{2}(z(\theta)+z(\theta)^{-1})}\frac{z'(\theta)}{iz(\theta)}\,\mathrm{d}\theta = \int_C \frac{z + z^{-1}}{10 - 4(z+z^{-1})}\frac{1}{iz}\,\mathrm{d}z.$$

We now observe that

$$\int_C \frac{z+z^{-1}}{10 - 4(z+z^{-1})}\frac{1}{iz}\,\mathrm{d}z = \frac{1}{i}\int_C \frac{z^2+1}{z(10z - 4z^2 - 4)}\,\mathrm{d}z = \frac{i}{2}\int_C \frac{z^2+1}{z(2z^2 - 5z + 2)}\,\mathrm{d}z.$$

Let

$$f(z) = \frac{z^2+1}{z(2z^2-5z+2)} = \frac{z^2+1}{z(2z-1)(z-2)}$$

so that $f(z)$ has simple poles at $z_0 = 0$, $z_1 = 1/2$, and $z_2 = 2$.

(continued)

Notice that only $z_0 = 0$ and $z_1 = 1/2$ are inside C. Therefore,

$$\operatorname{Res}(f(z); 0) = \left.\frac{z^2 + 1}{(2z-1)(z-2)}\right|_{z=0} = \frac{1}{2}$$

and

$$\operatorname{Res}(f(z); 1/2) = (z - 1/2)\left.\frac{z^2 + 1}{z(2z-1)(z-2)}\right|_{z=1/2} = \left.\frac{z^2+1}{2z(z-2)}\right|_{z=1/2} = \frac{5/4}{1/2 - 2} = -\frac{5}{6}.$$

By the residue theorem we obtain

$$\int_0^{2\pi} \frac{\cos\theta}{5 - 4\cos\theta}\,\mathrm{d}\theta = \frac{i}{2}\int_C f(z)\,\mathrm{d}z = \frac{i}{2} 2\pi i \left(\frac{1}{2} - \frac{5}{6}\right) = \frac{\pi}{3}.$$

6. (a) Consider $z^5 - 1 = 0$. The solutions of this equation are $z_0 = 1$, $z_1 = e^{2\pi i/5}$, $z_2 = e^{4\pi i/5}$, $z_3 = e^{6\pi i/5}$, and $z_4 = e^{8\pi i/5}$. Since $(z-1)(z^4 + z^3 + z^2 + z + 1) = (z^5 - 1)$, we conclude that the roots of $(z-1)(z^4 + z^3 + z^2 + z + 1)$ must equal the roots of $z^5 - 1$. Clearly, $z_0 = 1$ is the root of $(z - 1)$. This means that the four roots of $P(z) = z^4 + z^3 + z^2 + z + 1$ must be the other four roots of $z^5 - 1$, namely $z_1 = e^{2\pi i/5}$, $z_2 = e^{4\pi i/5}$, $z_3 = e^{6\pi i/5}$, and $z_4 = e^{8\pi i/5}$.

(b) Notice that we can write

$$f(z) = \frac{z^2 - z}{z^9 - z^4} = \frac{z(z-1)}{z^4(z^5-1)} = \frac{z(z-1)}{z^4(z-1)(z^4 + z^3 + z^2 + z + 1)} = \frac{z(z-1)}{z^4(z-1)P(z)}$$

which is a ratio of polynomials. This means that isolated singular points will occur precisely where the denominator is 0. Notice that $z^4(z-1)P(z)$ has 6 zeros, namely $z_0 = 1$, $z_1 = e^{2\pi i/5}$, $z_2 = e^{4\pi i/5}$, $z_3 = e^{6\pi i/5}$, $z_4 = e^{8\pi i/5}$, and $z_5 = 0$. Now consider the numerator, $z(z-1)$, which has zeros at $z_0 = 1$ and $z_5 = 0$. Since the order of the zero at $z_0 = 1$ is the same in both the numerator and the denominator, we conclude $z_0 = 1$ is a removable singularity. Since the order of the zero at $z_5 = 0$ is 1 in the numerator and 4 in the denominator, we conclude that $z_5 = 0$ is a pole of order $4 - 1 = 3$. Finally, since the zeros of $P(z)$ are not zeros of the numerator, we conclude that $z_1 = e^{2\pi i/5}$, $z_2 = e^{4\pi i/5}$, $z_3 = e^{6\pi i/5}$, and $z_4 = e^{8\pi i/5}$ are each simple poles.

7. Suppose that $f(z) = z^3 e^{-1/z}$. Observe that $z_0 = 0$ is an isolated singular point of $f(z)$ that lies inside C. Therefore, we conclude from the residue theorem that

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}(f(z); 0).$$

However, since $z_0 = 0$ is clearly an essential singularity, the only way to compute $\operatorname{Res}(f(z); 0)$ is to determine the Laurent series for $f(z)$ valid for $|z| > 0$. Now,

$$e^{-1/z} = 1 - \frac{1}{z} + \frac{1}{2!z^2} - \frac{1}{3!z^3} + \frac{1}{4!z^4} - \frac{1}{5!z^5} + \cdots$$

for $|z| > 0$ so that

$$z^3 e^{-1/z} = z^3 - z^2 + \frac{z}{2!} - \frac{1}{3!} + \frac{1}{4!z} - \frac{1}{5!z^2} + \cdots$$

for $|z| > 0$.

(continued)

Thus, $\mathrm{Res}(f(z);0) = \frac{1}{4!}$ so that

$$\int_C z^3 e^{-1/z}\,\mathrm{d}z = 2\pi i \frac{1}{4!} = \frac{\pi i}{12}.$$

8. (a) Observe that the Laurent series for $h(w) = \frac{\sin w}{w}$ about the point 0 is

$$\frac{\sin w}{w} = \frac{1}{w}\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} w^{2j+1} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} w^{2j} = 1 - \frac{w^2}{3!} + \frac{w^4}{5!} - \frac{w^6}{7!} + \cdots .$$

This tells us that $w = 0$ is a removable singularity for $h(w)$. Hence, in order for $g(w)$ to be analytic at $w = 0$, it must be the case that

$$g(0) = \lim_{w\to 0} \frac{\sin w}{w} = \lim_{w\to 0}\left(1 - \frac{w^2}{3!} + \frac{w^4}{5!} - \frac{w^6}{7!} + \cdots\right) = 1.$$

Since $g(0) = w_0$, we conclude that $w_0 = 1$.

(b) Observe that if $w \neq 0$, then $g(w) = 0$ if and only if $\sin w = 0$. Since $\sin w = 0$ if and only if $w = k\pi$ for some $k \in \mathbb{Z}$, we conclude that $g(w) = 0$ if and only if $w = k\pi$ for some $k \in \mathbb{Z} \setminus \{0\}$. Since $|k\pi| > 3$ for any $k \in \mathbb{Z} \setminus \{0\}$, we conclude that $g(w) \neq 0$ for any $|w| \leq 3$.

(c) Suppose that f is entire. Fix z with $|z| < 1$ and consider the function

$$F(\zeta) = \frac{f(\zeta)}{g(\zeta - z)}$$

defined for any $|\zeta| \leq 2$. As a result of **(b)**, we know that $F(\zeta)$ is analytic inside and on the unit circle C since $g(\zeta - z) \neq 0$ for any $|z| < 1$ and $|\zeta| \leq 2$. (Indeed, suppose that $|z| < 1$ and $|\zeta| \leq 2$. If $w = \zeta - z$, then by the triangle inequality $|\zeta - z| \leq |\zeta| + |z| \leq 2 + 1 = 3$. Thus from **(b)**, we have $g(\zeta - z) = g(w) \neq 0$.) Therefore, we can apply the Cauchy integral theorem to conclude

$$F(z) = \frac{1}{2\pi i}\int_C \frac{F(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

However, since $g(0) = 1$ by **(a)**, we find

$$F(z) = \frac{f(z)}{g(z-z)} = \frac{f(z)}{g(0)} = f(z).$$

Furthermore, if $|\zeta| \leq 2$ with $\zeta \neq z$, then

$$\frac{F(\zeta)}{\zeta - z} = \frac{f(\zeta)}{(\zeta - z)g(\zeta - z)} = \frac{f(\zeta)}{\sin(\zeta - z)}$$

so that

$$f(z) = F(z) = \frac{1}{2\pi i}\int_C \frac{F(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\sin(\zeta - z)}\,\mathrm{d}\zeta$$

as required.