Solutions to Math 305 Midterm Exam #1

1. (a) If $S \subseteq \mathbb{R}$ is a set, then a is the supremum of S if the following two conditions hold: (i) $a \geq s$ for every $s \in S$, and (ii) if $a' \geq s$ for all $s \in S$, then $a \leq a'$. The real number b is the infimum of S if the following two conditions hold: (i) $b \leq s$ for all $s \in S$ and (ii) if $b' \leq s$ for all $s \in S$, then $b \geq b'$.

1. (b) The completeness axiom states the following. If $S \subseteq \mathbb{R}$ is a nonempty and bounded set, then $\sup S$ exists as a real number.

1. (c) Let $b = \inf S$ so that $b \leq x$ for every $x \in S$. Therefore, $-5b \geq -5x$ for all $x \in S$ implying that $-5b \geq y$ for all $y \in T$. Thus, $-5b$ is an upper bound for T. To show that $-5b$ is the least upper bound (or supremum) of T we will show that $-5b \leq a'$ for any a' such that $a' \geq y$ for all $y \in T$. Consider such an a'. Since $a' \geq y$ for all $y \in T$, we know that $a' \geq -5x$ for all $x \in S$. That is, $-a'/5 \leq x$ for every $x \in S$. This shows that $-a'/5$ is a lower bound for S. Since b is the infimum of S we know that $-a'/5 \leq b$, or equivalently $a' \geq -5b$. Hence, $\sup T = -5b$, or equivalently, $\sup T = -5 \inf S$.

2. (a) The Heine-Borel Theorem states the following. A set $S \subseteq \mathbb{R}$ is compact if and only if S is closed and bounded.

2. (b) (Using the definition of compact.) In order to prove that $S \cup T$ is compact, we must show that any open cover of $S \cup T$ contains a finite subcover. Thus, suppose that $\mathcal{F}$ is an open cover of $S \cup T$. Consider the collections $\mathcal{S} = \mathcal{F} \cap S = \{F \cap S : F \in \mathcal{F}\}$ and $\mathcal{T} = \mathcal{F} \cap T = \{F \cap T : F \in \mathcal{F}\}$ so that $\mathcal{S}$ is an open cover of S and $\mathcal{T}$ is an open cover of T. Since S is compact, there is a finite subcover of $\mathcal{S}$, call it $\mathcal{S}_0$, that covers S. Since T is compact, there is a finite subcover of $\mathcal{T}$, call it $\mathcal{T}_0$, that covers T. Therefore, the collection $\mathcal{S}_0 \cup \mathcal{T}_0$ is a subcover of $\mathcal{F}$ which is also a finite cover of $S \cup T$ (since the union of a finite number of objects is finite). Hence, any open cover of $S \cup T$ contains a finite subcover proving that $S \cup T$ is compact.

2. (b) (Using the Heine-Borel Theorem.) In order to prove that $S \cup T$ is compact, we must show that $S \cup T$ is closed and bounded. Since S is compact, we know that S is closed and bounded, and since T is compact we know that T is closed and bounded. In order to show that $S \cup T$ is bounded, we need to show that there exists some $N \in \mathbb{N}$ such that $|x| \leq N$ for all $x \in S \cup T$. Since S is bounded, we know that there exists some $n \in \mathbb{N}$ such that $|s| \leq n$ for all $n \in S$, and since T is bounded, we know that there exists some $m \in \mathbb{N}$ such that $|t| \leq m$ for all $t \in T$. Therefore, if we set $N = m + n$ and let $x \in S \cup T$, then either $x \in S$ in which case $x \leq m < N$ or $x \in T$ in which case $x \leq n < N$. If it happens that $x \in S \cap T$, then $|x| \leq \max\{n, m\} < N$. In any case, we see that $|x| \leq N$ proving that $S \cup T$ is bounded. To show that $S \cup T$ is closed, we need to prove that the union of two closed sets is closed. Equivalently, we need to prove that $(S \cup T)^c = S^c \cap T^c$ is open. Since S^c is open, we know that if $s \in S^c$, then there exists an ε_1 such that $N(s; \varepsilon_1) \subseteq S^c$, and since T^c is open, we know that if $t \in T^c$, then there exists an ε_2 such that $N(x; \varepsilon_2) \subseteq S^c$. Hence, suppose that $x \in S^c \cap T^c$ and let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ so that $N(x; \varepsilon) \subseteq N(x; \varepsilon_1) \subseteq S^c$ and $N(x; \varepsilon) \subseteq N(x; \varepsilon_2) \subseteq T^c$ which implies that $N(x; \varepsilon) \subseteq S^c \cap T^c$. This implies that $S^c \cap T^c$ is open so that $(S^c \cap T^c)^c = S \cup T$ is closed.

3. (a) To show that f is not bijective, it is sufficient to show that there exist points $x_1 \in [-2,2]$ and $x_2 \in [-2,2]$ with $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. If we take $x_1 = -1$ and $x_2 = 1$, then $x_1 \neq x_2$ but $f(x_1) = f(x_2) = 1$. Hence, f is not bijective.

3. (b) To show that $f^{-1}(S)$ is an open set, it is sufficient to show that if $x \in f^{-1}(S)$, then there exists some $\varepsilon > 0$ such that $N(x;\varepsilon) \subseteq f^{-1}(S)$. Therefore, let $x \in f^{-1}(S)$ so that $x^2 \in S$. Consider $N(x;\varepsilon) = (x-\varepsilon, x+\varepsilon)$. Since $f(x) = x^2$, we know that $f(N(x;\varepsilon)) = f((x-\varepsilon, x+\varepsilon)) = ((x-\varepsilon)^2, (x+\varepsilon)^2)$. Since $x^2 \in S$ and S is open we know that there exists some $\varepsilon_1 > 0$ such that $N(x^2;\varepsilon_1) = (x^2-\varepsilon_1, x^2+\varepsilon_1) \subseteq S$. Hence, the proof will be completed if we can choose ε such that $((x-\varepsilon)^2, (x+\varepsilon)^2) \subseteq x^2 + \varepsilon_1$. Observe that $(x+\varepsilon)^2 = x^2 + 2x\varepsilon + \varepsilon^2$ and so we choose ε such that $2x\varepsilon + \varepsilon^2 < \varepsilon_1$.

3. (c) Observe that $f^{-1}(T^c) = \{x \in [-2,2] : f(x) \in T^c\} = \{x \in [-2,2] : f(x) \notin T\}$. However, the set of $x \in [-2,2]$ such that $f(x) \notin T$ is, by definition, the complement of the set of $x \in [-2,2]$ such that $f(x) \in T$. Therefore, we find

$$f^{-1}(T^c) = \{x \in [-2,2] : f(x) \notin T\} = \{x \in [-2,2] : f(x) \in T\}^c = [f^{-1}(T)]^c$$

as required.

4. We claim that

$$\operatorname{bd} S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}.$$

Suppose first that $x = 0$ and let $\varepsilon > 0$ be arbitrary. The Archimedean property implies that there exists some $m \in \mathbb{N}$ such that $0 < 1/m < \varepsilon$. Moreover, since $\mathbb{R}$ is complete, we know that there exists some $y \in \mathbb{R}$ such that $0 < 1/(m+1) < y < 1/m < \varepsilon$. Thus, $y \in S$ and $y \in N(0;\varepsilon)$ so that $N(x;\varepsilon) \cap S \neq \emptyset$. Since $0 \notin S$ we conclude that $N(0;\varepsilon) \cap S^c \neq \emptyset$ so that $0 \in \operatorname{bd} S$. Now assume that $x = 1/n$ for some $n \in \mathbb{N}$ and let $\varepsilon > 0$ be arbitrary. Since $1/n \notin S$ we conclude that $N(1/n;\varepsilon) \cap S^c \neq \emptyset$. Since $\mathbb{R}$ is complete, we know that if there exists some irrational y with

$$y \in \left\{x \in \mathbb{R} : \frac{1}{n} - \varepsilon < x < \frac{1}{n} + \varepsilon\right\} = \left[\frac{1}{n} - \varepsilon, \frac{1}{n} + \varepsilon\right]$$

for every $\varepsilon > 0$. Thus, $y \in S$ and $y \in N(1/n;\varepsilon)$ so that $N(1/n;\varepsilon) \cap S \neq \emptyset$ for every $\varepsilon > 0$. Hence, we have shown that

$$\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\} \subseteq \operatorname{bd} S.$$

To show the reverse containment, suppose that $x \in (0,1)$ with $x \neq 1/n$ for some $n \in \mathbb{N}$. The Archimedean property implies that there exists some $m \in \mathbb{N}$ such that $1/(m+1) < x < 1/m$. If we set

$$\varepsilon = \frac{1}{2}\left(\frac{1}{m} - \frac{1}{m+1}\right) = \frac{1}{2m(m+1)},$$

then

$$N(x;\varepsilon) \cap \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\} = \emptyset.$$

Thus, $x \notin \operatorname{bd} S$. If $x < 0$, let $\varepsilon = -x/2$ so that $N(x;\varepsilon) \cap (0,1) = \emptyset$ so that $x \notin \operatorname{bd} S$. If $x > 1$, let $\varepsilon = (x-1)/2$ so that $N(x;\varepsilon) \cap (0,1) = \emptyset$ so that $x \notin \operatorname{bd} S$. Thus we have shown that

$$\operatorname{bd} S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}.$$

5. We claim that $\operatorname{bd} S = [0,1]$. Suppose that

$$y \in \left\{ x \in \mathbb{R} : \frac{1}{n+1} \leq x \leq \frac{1}{n} \right\} = \left[\frac{1}{n+1}, \frac{1}{n} \right].$$

For any $\varepsilon > 0$, we know that $N(y;\varepsilon)$ contains irrational numbers so that $N(y;\varepsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$. But we know that $N(y;\varepsilon)$ also contains some $r \in \mathbb{Q}$ such that $1/(n+1) < r < 1/n$. Thus,

$$\left[\frac{1}{n+1}, \frac{1}{n} \right] \subseteq \operatorname{bd} S$$

for every $n \in \mathbb{N}$. Observe $0 \notin S$ so that we cannot immediately conclude that $N(0;\varepsilon) \cap S \neq \emptyset$. However, we knows that if $\varepsilon > 0$, then there exists some n such that $0 < 1/n < \varepsilon$ which implies that

$$N(0;\varepsilon) \cap \left[\frac{1}{n+1}, \frac{1}{n} \right] \neq \emptyset.$$

Thus, $N(0;\varepsilon) \cap S \neq \emptyset$ so that $0 \in \operatorname{bd} S$. Since we can write

$$[0,1] = \{0\} \cup \bigcup_{n \in \mathbb{N}} \left[\frac{1}{n+1}, \frac{1}{n} \right],$$

we conclude that $[0,1] \subseteq \operatorname{bd} S$. To show that $[0,1] = \operatorname{bd} S$, suppose that $y \notin [0,1]$ so that either $y < 0$ or $y > 1$. If $y < 0$, let $\varepsilon = -y/2$ so that $N(y;\varepsilon) \cap [0,1] = \emptyset$ so that $y \notin \operatorname{bd} S$. If $y > 1$, let $\varepsilon = (y-1)/2$ so that $N(y;\varepsilon) \cap [0,1] = \emptyset$ so that $y \notin \operatorname{bd} S$. Thus, if $y \notin [0,1]$, then $y \notin \operatorname{bd} S$. This implies that $\operatorname{bd} S = [0,1]$.

6. We claim that $S' = \{-1/2, 1/2\}$ so that $\operatorname{cl} S = S \cup S'$. Observe that

$$\left\{ \frac{n}{2n+1} : n \in \mathbb{N} \right\} = \left\{ \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots \right\}$$

so that the terms get arbitrarily close to $1/2$. Therefore,

$$\left\{ (-1)^n \frac{n}{2n+1} : n \in \mathbb{N} \right\} = \left\{ -\frac{1}{3}, \frac{2}{5}, -\frac{3}{7}, \frac{4}{9}, \ldots \right\} = \left\{ -\frac{1}{3}, -\frac{3}{7}, \ldots \right\} \cup \left\{ \frac{2}{5}, \frac{4}{9}, \ldots \right\}.$$

The terms in

$$\left\{ -\frac{1}{3}, -\frac{3}{7}, \ldots \right\}$$

get arbitrarily close to $-1/2$ while the terms in

$$\left\{ \frac{2}{5}, \frac{4}{9}, \ldots \right\}$$

get arbitrarily close to $1/2$. Hence, we find $S' = \{-1/2, 1/2\}$. To prove that this is indeed S', we know from the Archimedean property that for every $\varepsilon > 0$ there exists an n such that

$$\frac{1}{2} - \varepsilon < \frac{n}{2n+1} < \frac{1}{2} \quad \text{and} \quad -\frac{1}{2} < -\frac{n}{2n+1} < -\frac{1}{2} + \varepsilon.$$

This means that for every $\varepsilon > 0$ we have $N^*(1/2, \varepsilon) \cap S \neq \emptyset$ and $N^*(-1/2, \varepsilon) \cap S \neq \emptyset$ implying that $\{-1/2, 1/2\} \subseteq S$. On the other hand, if $x > 1/2$ or $x < -1/2$ and $\varepsilon = (|x| - 1/2)/2$, then $N^*(1/2, \varepsilon) \cap S = \emptyset$ and $N^*(-1/2, \varepsilon) \cap S = \emptyset$. Similarly, if $-1/3 < x < 2/5$ and $\varepsilon = \max\{|2/5 - x|, |-1/3 - x|\}/2$, then $N^*(x, \varepsilon) \cap S = \emptyset$. Finally, if $2/5 \leq x < 1/2$, then by the Archimedean property, there exists an n such that

$$\frac{n}{2n+1} < x < \frac{n+2}{2(n+2)+1},$$

while if $-1/2 < x < -1/3$, then by the Archimedean property, there exists an n such that

$$-\frac{n+2}{2(n+2)+1} < x < -\frac{n}{2n+1}.$$

This implies that $S' = \{-1/2, 1/2\}$.