Math 305 Midterm Exam #1 – October 13, 2011

This exam is worth 60 points.

This exam has 6 problems and 1 numbered page.

You have 75 minutes to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements. For problems with multiple parts, all parts are equally weighted.*

This exam is closed book. There are no aids whatsoever allowed.

Instructor: Michael Kozdron

Some Notation

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers.
- $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ denotes the set of integers.
- $\mathbb{Q} = \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\right\}$ denotes the set of rational numbers.
- $\mathbb{R}$ denotes the set of real numbers.
- If $A \subseteq \mathbb{R}$ is a set, then the complement of A is defined by
$$A^c = \mathbb{R} \setminus A = \{x \in \mathbb{R} : x \notin A\}.$$
- Let $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ be sets. If $f : A \to B$ is a function and $C \subseteq B$, then the set $f^{-1}(C)$ is defined by
$$f^{-1}(C) = \{x \in A : f(x) \in C\}.$$

1. (*15 points*)

(a) Let $S \subseteq \mathbb{R}$ be a set. Define what it means for the real number a to be the supremum of S, and define what it means for the real number b to be the infimum of S.

(b) State the completeness axiom for $\mathbb{R}$.

(c) Suppose that $S \subseteq \mathbb{R}$ is nonempty and bounded. Define the set T to be

$$T = \{y \in \mathbb{R} : y = -5x \text{ for some } x \in S\}.$$

Prove that $\sup T = -5 \inf S$.

2. (*12 points*)

(a) State the Heine-Borel Theorem.

(b) Suppose that S and T are both compact subsets of $\mathbb{R}$. Prove that their union $S \cup T$ is compact. *You may answer this question using either the definition of compact set or the characterization of compactness given by the Heine-Borel Theorem.*

3. (*15 points*) Consider the function $f : [-2, 2] \to [0, 4]$ given by $f(x) = x^2$.

(a) Prove that f is not bijective.

(b) Suppose that $S \subseteq [0, 4]$ is an open set. Prove that $f^{-1}(S) \subseteq [-2, 2]$ is an open set.

(c) Suppose that $T \subseteq [0, 4]$ is a nonempty set. Prove that $f^{-1}(T^c) = [f^{-1}(T)]^c$.

4. (*6 points*) Consider the set $S \subseteq \mathbb{R}$ given by

$$S = \bigcup_{n=1}^{\infty} \left\{ x \in \mathbb{R} : \frac{1}{n+1} < x < \frac{1}{n} \right\}.$$

Determine bd S, the set of boundary points of S, and prove your answer.

5. (*6 points*) Consider the set $S \subseteq \mathbb{R}$ given by

$$S = \bigcup_{n=1}^{\infty} \left\{ x \in \mathbb{Q} : \frac{1}{n+1} < x < \frac{1}{n} \right\}.$$

Determine bd S, the set of boundary points of S, and prove your answer.

6. (*6 points*) Consider the set $S \subseteq \mathbb{R}$ given by

$$S = \left\{ (-1)^n \frac{n}{2n+1} : n \in \mathbb{N} \right\}.$$

Determine cl S, the closure of S, and prove your answer.