Mathematics 305 Fall 2011 Final Exam – Solutions

1. Suppose that $E \subseteq \mathbb{R}$ is nonempty and let $x \in \mathbb{R}$.

(a) We say $x \in \operatorname{int} E$ if there exists some $\varepsilon > 0$ such that $N(x;\varepsilon) \cap E^c = \emptyset$.

(b) We say $x \in \operatorname{bd} E$ if $N(x;\varepsilon) \cap E \neq \emptyset$ and $N(x;\varepsilon) \cap E^c \neq \emptyset$ for every $\varepsilon > 0$.

(c) We say $x \in \operatorname{cl} E$ if $N(x;\varepsilon) \cap E \neq \emptyset$ for every $\varepsilon > 0$.

(d) We will establish this result by showing two set containments. To begin, suppose that $x \in (\operatorname{cl} E) \setminus (\operatorname{int} E)$ so that $x \in (\operatorname{cl} E) \cap (\operatorname{int} E)^c$. Since $x \notin \operatorname{int} E$ there does not exist any $\varepsilon > 0$ such that $N(x;\varepsilon) \cap E^c = \emptyset$. This is logically equivalent to saying that $N(x;\varepsilon) \cap E^c \neq \emptyset$ for every $\varepsilon > 0$. Since $x \in \operatorname{cl} E$, we know $N(x;\varepsilon) \cap E \neq \emptyset$ for every $\varepsilon > 0$. Hence, $N(x;\varepsilon) \cap E^c \neq \emptyset$ for every $\varepsilon > 0$ and $N(x;\varepsilon) \cap E \neq \emptyset$ for every $\varepsilon > 0$ so that by definition $x \in \operatorname{bd} E$. Conversely, suppose that $x \in \operatorname{bd} E$ so that $N(x;\varepsilon) \cap E^c \neq \emptyset$ for every $\varepsilon > 0$ and $N(x;\varepsilon) \cap E \neq \emptyset$ for every $\varepsilon > 0$. Since $N(x;\varepsilon) \cap E \neq \emptyset$ for every $\varepsilon > 0$, we conclude by definition that $x \in \operatorname{cl} E$. Since $N(x;\varepsilon) \cap E^c \neq \emptyset$ for every $\varepsilon > 0$ we conclude that there does not exist any $\varepsilon > 0$ such that $N(x;\varepsilon) \cap E^c = \emptyset$. Thus, by definition, $x \notin \operatorname{int} E$ so that $x \in (\operatorname{int} E)^c$. Therefore, $x \in (\operatorname{cl} E) \cap (\operatorname{int} E)^c = (\operatorname{cl} E) \setminus (\operatorname{int} E)$.

2. There does not exist a subset $E \subseteq \mathbb{Q}$ such that $\operatorname{cl} E = [0,1] \cup \{\sqrt{2}\}$. Here is the proof. Suppose that $E \subseteq \mathbb{Q}$. By definition, $\operatorname{cl} E = E \cup E'$ where E' is the set of accumulation points of E. Since $\sqrt{2} \notin \mathbb{Q}$, in order for $\sqrt{2}$ to be a point of $\operatorname{cl} E$, it is necessarily the case that $\sqrt{2} \in E'$. By definition, $\sqrt{2} \in E'$ if and only if $N^*(\sqrt{2};\varepsilon) \cap E \neq \emptyset$ for every $\varepsilon > 0$. In particular, choosing $\varepsilon = 1/2$ implies that $N^*(\sqrt{2};\varepsilon) = (\sqrt{2} - 1/2, \sqrt{2}) \cup (\sqrt{2}, \sqrt{2} + 1/2)$ contains a point of E. Call this point y. However, if $y \in E$, then trivially $y \in \operatorname{cl} E$. However, by construction, $y \in \mathbb{Q}$ with $y > 1$ so that $y \notin [0,1] \cup \{\sqrt{2}\} = \operatorname{cl} E$. This contradicts the fact that $y \in \operatorname{cl} E$ and proves no such $E \subseteq \mathbb{Q}$ can exist.

3. In order to prove this result, we must establish two implications. For the first implication, suppose that $x \in \operatorname{cl} S$ so that $x \in S \cup S'$. If $x \in S$, then the sequence $\{x_n\}$ defined by $x_n = x$ for all $n \in \mathbb{N}$ trivially converges to x. Now suppose that $x \in S'$ so that $N^*(x;\varepsilon) \cap S \neq \emptyset$ for every $\varepsilon > 0$. Let $\varepsilon_n = 1/n$ for $n \in \mathbb{N}$ and choose $x_n \in N^*(x;\varepsilon_n) \cap S$ which is possible by the assumption $x \in S'$. The sequence $\{x_n\}$ converges to x by the squeeze theorem since

$$|x_n - x| < \varepsilon_n = \frac{1}{n}$$

and $1/n \to 0$ as $n \to \infty$. Hence, $x \in \operatorname{cl} S$ implies there exists a sequence $\{x_n\}$ with $x_n \in S$ such that $x_n \to x$. To prove the reverse implication, suppose that there exists a sequence $\{x_n\}$ with $x_n \in S$ such that $x_n \to x$. To show that $x \in \operatorname{cl} S$, we must show that $N(x;\varepsilon) \cap S \neq \emptyset$ for every $\varepsilon > 0$. Since $x_n \to x$, we know that for every $\varepsilon > 0$ there exists an N such that $n > N$ implies

$$|x_n - x| < \varepsilon.$$

In particular, (using $n = N + 1$) we know that $|x_{N+1} - x| < \varepsilon$. However, since $N(x;\varepsilon) = \{y : |x - y| < \varepsilon\}$, we know that $x_{N+1} \in N(x;\varepsilon)$. By assumption, $x_{N+1} \in S$ and so $x_{N+1} \in N(x;\varepsilon) \cap S$. As $\varepsilon > 0$ was arbitrary, this shows that if there exists a sequence $\{x_n\}$ with $x_n \in S$ such that $x_n \to x$, then $x \in \operatorname{cl} S$.

4. Notice that $S = S_1 \cup S_2 \cup S_3 \cup S_4$ where

$$\begin{aligned} S_1 = \left\{ a_n + (-1)^n + \frac{1}{n} : n = 1, 5, 9, \ldots \right\} &= \left\{ 1 + (-1) + \frac{1}{n} : n = 1, 5, 9, \ldots \right\} \\ &= \left\{ \frac{1}{n} : n = 1, 5, 9, \ldots \right\} \\ &= \left\{ 1, \frac{1}{5}, \frac{1}{9}, \ldots \right\}, \end{aligned}$$

$$\begin{aligned} S_2 = \left\{ a_n + (-1)^n + \frac{1}{n} : n = 2, 6, 10, \ldots \right\} &= \left\{ 2 + (+1) + \frac{1}{n} : n = 2, 6, 10, \ldots \right\} \\ &= \left\{ 3 + \frac{1}{n} : n = 2, 6, 10, \ldots \right\} \\ &= \left\{ 3\frac{1}{2}, 3\frac{1}{6}, 3\frac{1}{10}, \ldots \right\}, \end{aligned}$$

$$\begin{aligned} S_3 = \left\{ a_n + (-1)^n + \frac{1}{n} : n = 3, 7, 11, \ldots \right\} &= \left\{ 3 + (-1) + \frac{1}{n} : n = 3, 7, 11, \ldots \right\} \\ &= \left\{ 2 + \frac{1}{n} : n = 3, 7, 11, \ldots \right\} \\ &= \left\{ 2\frac{1}{3}, 2\frac{1}{7}, 2\frac{1}{11}, \ldots \right\}, \end{aligned}$$

and

$$\begin{aligned} S_4 = \left\{ a_n + (-1)^n + \frac{1}{n} : n = 4, 8, 12, \ldots \right\} &= \left\{ 4 + (+1) + \frac{1}{n} : n = 4, 8, 12, \ldots \right\} \\ &= \left\{ 5 + \frac{1}{n} : n = 4, 8, 12, \ldots \right\} \\ &= \left\{ 5\frac{1}{4}, 5\frac{1}{8}, 5\frac{1}{12}, \ldots \right\}. \end{aligned}$$

Observe that each of S_1, S_2, S_3, and S_4 defines a strictly decreasing sequence. Therefore, $\sup S_i = \max S_i$ for each $i = 1, 2, 3, 4$ and $\min S_i$ does not exist for $i = 1, 2, 3, 4$. In particular,

$$\sup S_1 = \max S_1 = 1, \quad \sup S_2 = \max S_2 = 3\frac{1}{2},$$

$$\sup S_3 = \max S_3 = 2\frac{1}{3}, \quad \sup S_4 = \max S_4 = 5\frac{1}{4}$$

and

$$\inf S_1 = 0, \quad \inf S_2 = 3, \quad \inf S_3 = 2, \quad \inf S_4 = 5.$$

(*continued*)

(a) We find

$$\sup S = \sup\{S_1 \cup S_2 \cup S_3 \cup S_4\} = \sup\left\{1, 3\frac{1}{2}, 2\frac{1}{3}, 5\frac{1}{4}\right\} = 5\frac{1}{4}$$

and

$$\inf S = \inf\{S_1 \cup S_2 \cup S_3 \cup S_4\} = \inf\{0, 3, 2, 5\} = 0.$$

(b) Since $5\frac{1}{4} \in S$ and $0 \notin S$, we conclude

$$\max S = 5\frac{1}{4} \quad \text{and} \quad \min S \text{ does not exist.}$$

(c) There are 4 accumulation points of S, namely the limit points of S_i for $i = 1, 2, 3, 4$. Therefore, $S' = \{0, 2, 3, 5\}$ so that

$$\operatorname{cl} S = S \cup S' = S \cup \{0, 2, 3, 5\}.$$

5. Observe that $a_2 = 7/4$ and $a_3 = 17/8$. Therefore, assume that $n \geq 1$ and note that $a_{n+1} \geq a_n$ if and only if

$$\frac{1}{4}(2a_n + 5) \geq a_n$$

which holds if and only if $2a_n + 5 \geq 4a_n$ which in turn holds if and only if $2a_n \leq 5$ or, equivalently, if and only if $a_n \leq 5/2$. We will now show that if $n \geq 1$, then $a_n \leq 5/2$ which will therefore imply that a_n is bounded above and increasing for $n \geq 1$. Observe that $a_1 \leq 5/2$ and $a_2 \leq 5/2$. Therefore, assume that $a_n \leq 5/2$ for some $n \geq 1$. Hence,

$$a_{n+1} = \frac{1}{4}(2a_n + 5) \leq \frac{1}{4}\left(2 \cdot \frac{5}{2} + 5\right) = \frac{10}{4} = \frac{5}{2}.$$

Thus, by induction, $a_n \leq 5/2$ for all $n \geq 1$ and so $\{a_n\}$ is necessarily increasing. Since $\{a_n\}$ is bounded above and increasing, we know $\{a_n\}$ must converge. Therefore, if $a = \lim a_n$, then a satisfies

$$a = \frac{1}{4}(2a + 5) \quad \text{or, equivalently,} \quad 4a = 2a + 5 \quad \text{and so} \quad a = \frac{5}{2}.$$

6. In order to prove that $\{a_n\}$ converges it is sufficient to prove that $\{a_n\}$ is bounded since assumption (ii) tells us that $\{a_n\}$ is increasing and we know that a bounded, increasing sequence is necessarily convergent. Note that assumption (i) tells us that $|a_n| = a_n$ for all n. Suppose that $\{a_{n_k}\}$ is a convergent subsequence of $\{a_n\}$ which exists by assumption (iii). This implies that $\{a_{n_k}\}$ is bounded. Thus, let M be such that $a_{n_k} < M$ for all n_k. We will now show that $a_n < M$ for all n. By assumption, $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$. This means that there are elements in $\{a_n\}$ which are not elements of $\{a_{n_k}\}$. Let $a_j \in \{a_n\} \setminus \{a_{n_k}\}$ be one of those elements. By assumption (ii), $\{a_{n_k}\}$ is an increasing sequence and so there exist elements a_{n_j} and $a_{n_{j+1}}$ such that $a_{n_j} \leq a_j \leq a_{n_{j+1}}$. But $a_{n_{j+1}} < M$ which implies that $a_j < M$. Thus, $\{a_n\}$ is bounded, increasing, and therefore convergent.

7. **(a)** Since f is defined on the closed interval $[-1, 2]$, it is necessarily the case that $|f(2)| < \infty$. Therefore, set $M = |f(2)|$ and note that by the triangle inequality combined with the inequality stated in the problem that

$$|f(x)| - |f(2)| \le |f(x) - f(2)| \le 5|x - 2|.$$

However, if $x \in [-1, 2]$, then $|x - 2| \le 3$, and so we conclude that

$$|f(x)| \le 5|x - 2| + |f(2)| \le 5(3) + M = 15 + M < \infty.$$

Thus, f is bounded on $[-1, 2]$ as required.

(b) Since $x_n \to 1$, we know that for every $\varepsilon > 0$ there exists some N such that $n > N$ implies $|x_n - 1| < \frac{\varepsilon}{5}$. Using the inequality stated in the problem,

$$|f(x_n) - f(1)| \le 5|x_n - 1| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon$$

so that $f(x_n) \to 1$ as $n \to \infty$ as required.

(c) (Solution 1) By part (a) we know that f is bounded. Therefore, $f(x_n)$ has a convergent subsequence.

(c) (Solution 2) Since the sequence $\{x_n\}$ is bounded, it is necessarily the case that there exists some subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges. Choose this convergent subsequence $\{x_{n_k}\}$ so that the required sequence of integers is $\{n_k\}$. In order to prove that $\{f(x_{n_k})\}$ converges, we will prove that it is a Cauchy sequence. Since $\{x_{n_k}\}$ converges, it is necessarily a Cauchy sequence. This means that for every $\varepsilon > 0$, there exists some N such that $n_k, n_j > N$ implies

$$|x_{n_k} - x_{n_j}| < \frac{\varepsilon}{5}.$$

Hence, using the inequality stated in the problem,

$$|f(x_{n_k}) - f(x_{n_j})| \le 5|x_{n_k} - x_{n_j}| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon,$$

and so $\{f(x_{n_k})\}$ is a Cauchy, and therefore convergent, sequence.

8. Note that

$$|2x^2 - 3x + 5 - 10| = |2x^2 - 3x - 5| = |x + 1||2x - 5|.$$

Hence, if $|x + 1| < 1$, then $|x| = |x + 1 - 1| \le |x + 1| + 1 < 1 + 1 = 2$ and so $|2x - 5| \le 2|x| + 5 < 2(2) + 5 = 9$. Therefore, if $\varepsilon > 0$ is given and $\delta = \min\{1, \varepsilon/9\}$, then

$$|2x^2 - 3x + 5 - 10| = |x + 1||2x - 5| < \varepsilon$$

whenever $|x + 1| < \delta$.

9. Notice that

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}.$$

By definition, f is continuous at 4 if

$$f(4) = \lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}.$$

Hence, we should define $f(4) = 1/4$.