Math 305 Final Exam – Thursday, December 15, 2011

This exam is worth 100 points.

This exam has 9 problems and 2 numbered page.

You have 3 hours to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements. For problems with multiple parts, all parts are equally weighted.*

This exam is closed book. There are no aids whatsoever allowed.

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Some Notation

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers.
- $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ denotes the set of integers.
- $\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0 \right\}$ denotes the set of rational numbers.
- $\mathbb{R}$ denotes the set of real numbers.
- If $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ are sets, then the set difference $A \setminus B$ is defined by
$$A \setminus B = A \cap B^c.$$
- If $A \subseteq \mathbb{R}$ is a set, then the complement of A is defined by
$$A^c = \mathbb{R} \setminus A = \{x \in \mathbb{R} : x \notin A\}.$$
- If $x \in \mathbb{R}$, then the ε-neighbourhood of x is $N(x; \varepsilon) = \{y \in \mathbb{R} : |x - y| < \varepsilon\}$ and the deleted ε-neighbourhood of x is $N^*(x; \varepsilon) = \{y \in \mathbb{R} : 0 < |x - y| < \varepsilon\}$.

1. (*16 points*) Let $E \subseteq \mathbb{R}$ be a nonempty set.

(a) Define what it means for $x \in \mathbb{R}$ to be an interior point of E.

(b) Define what it means for $x \in \mathbb{R}$ to be a boundary point of E.

(c) Define what it means for $x \in \mathbb{R}$ to be in the closure of E.

(d) Prove that $(\operatorname{cl} E) \setminus (\operatorname{int} E) = \operatorname{bd} E$.

2. (*9 points*) Does there exist a subset $E \subseteq \mathbb{Q}$ such that $\operatorname{cl} E = [0,1] \cup \{\sqrt{2}\}$? Explain.

3. (*9 points*) Suppose that $S \subseteq \mathbb{R}$. Prove that $x \in \operatorname{cl} S$ if and only if there exists a sequence $\{x_n\}$ with $x_n \in S$ for all $n \in \mathbb{N}$ such that $x_n \to x$.

4. (*15 points*) Let $\{a_n\}$ be the sequence defined by

$$a_n = \begin{cases} 1, & \text{if } n = 1, 5, 9, \ldots, \\ 2, & \text{if } n = 2, 6, 10, \ldots, \\ 3, & \text{if } n = 3, 7, 11, \ldots, \\ 4, & \text{if } n = 4, 8, 12, \ldots. \end{cases}$$

Define the set S by

$$S = \left\{ a_n + (-1)^n + \frac{1}{n} : n = 1, 2, 3, \ldots \right\}.$$

(a) Find $\sup S$ and $\inf S$.

(b) Do $\max S$ and $\min S$ exist? If so, find them.

(c) Find the set of accumulation points of S and determine the closure of S.

5. (*8 points*) Consider the sequence $\{a_n\}$ defined for $n \in \mathbb{N}$ by $a_1 = 1$ and

$$a_{n+1} = \frac{1}{4}(2a_n + 5).$$

Prove that $\{a_n\}$ converges and determine $\lim_{n \to \infty} a_n$.

6. (*8 points*) Suppose that $\{a_n\}$ is a sequence with the properties that

(i) $a_n > 0$ for every $n \in \mathbb{N}$,

(ii) $a_{n+1} \geq a_n$ for every $n \in \mathbb{N}$, and

(iii) $\{a_n\}$ has a convergent subsequence.

Prove that $\{a_n\}$ converges.

7. (*18 points*) Suppose that the function f is defined on the closed interval $[-1, 2]$ and has the property that

$$|f(x) - f(y)| \leq 5|x - y|$$

for all x and y in $[-1, 2]$.

(a) Prove that f is bounded on $[-1, 2]$.

(b) If $\{x_n\}$ is a sequence in $[-1, 2]$ with $\lim_{n\to\infty} x_n = 1$, prove that $\lim_{n\to\infty} f(x_n) = f(1)$.

(c) If $\{x_n\}$ is a sequence in $[-1, 2]$, show that there exists a sequence of integers $\{n_k\}$ such that $\{f(x_{n_k})\}$ converges.

8. (*9 points*) Use the ε-δ definition of limit to prove that $\lim_{x\to -1}(2x^2 - 3x + 5) = 10$.

9. (*8 points*) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x) = \frac{\sqrt{x} - 2}{x - 4}$$

for $x \neq 4$. How should $f(4)$ be defined so that f is continuous at 4? Explain.