Math 305 Fall 2011
Solutions to Assignment #1

2. To prove that $(A^c)^c = A$, we need to verify the two containments $(A^c)^c \subseteq A$ and $A \subseteq (A^c)^c$. We will begin by showing that $(A^c)^c \subseteq A$. Suppose that $x \in (A^c)^c$. By definition of complement, this means that $x \notin (A^c)$. But this says precisely that x is not in A^c which, by the definition of complement again, means exactly that x is in A. In other words, $x \in A$. To show the containment $A \subseteq (A^c)^c$, assume that $x \in A$. By the definition of complement, this means that x is not in A^c. In other words, $x \notin A^c$ so that $x \in (A^c)^c$.

3. **(a)** The proof of the distribution law $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ is outlined in Practice Problem 5.14 on page 43. The solution written out in full detail is on page 46.

(b) One proof that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ is to follow the same strategy as in **(a)** by showing the two required containments. Another proof that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ can be given using de Morgan's law for two sets (as proved in class on September 8, 2011, or see Problem #5 below) and the result of Problem #2. That is, if we replace A by A^c and B by B^c and C by C^c in part **(a)**, then we obtain

$$A^c \cup (B^c \cap C^c) = (A^c \cup B^c) \cap (A^c \cup C^c).$$

Taking complements of both sides gives

$$[A^c \cup (B^c \cap C^c)]^c = [(A^c \cup B^c) \cap (A^c \cup C^c)]^c$$

which by de Morgan's law is equivalent to

$$(A^c)^c \cap (B^c \cap C^c)^c = (A^c \cup B^c)^c \cup (A^c \cup C^c)^c.$$

Using de Morgan's law three more times shows this is equivalent to

$$(A^c)^c \cap [(B^c)^c \cup (C^c)^c] = [(A^c)^c \cap (B^c)^c] \cup [(A^c)^c \cap (C^c)^c].$$

Finally, Problem #2 implies this is equivalent to

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

as required.

4. Recall that the definition of $A \setminus B$ as given in class was

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

Notice, however, that this is exactly the same as the set $A \cap B^c$. Therefore,

$$(A \setminus B) \cup (A \cap B) \cup (B \setminus A) = (A \cap B^c) \cup (A \cap B) \cup (B \cap A^c).$$

We are now going to use the distribution law twice. Observe first that

$$(A \cap B^c) \cup (A \cap B) = A \cap (B \cup B^c) = A.$$

Thus, we can substitute this expression into the previous expression to conclude that

$$(A \cap B^c) \cup (A \cap B) \cup (B \cap A^c) = A \cup (B \cap A^c).$$

The distribution law now implies that

$$A \cup (B \cap A^c) = (A \cup B) \cap (A \cup A^c) = A \cup B.$$

In summary, we have shown that

$$(A \setminus B) \cup (A \cap B) \cup (B \setminus A) = A \cup B$$

as required.

5. In order to prove that

$$\left(\bigcup_{j \in J} A_j \right)^c = \bigcap_{j \in J} (A_j^c)$$

we will show the two separate containments. To begin, suppose that

$$x \in \left(\bigcup_{j \in J} A_j \right)^c$$

so that by the definition of complement we conclude that

$$x \notin \bigcup_{j \in J} A_j.$$

But this is the same as saying that x does not below to *any one* of the sets A_j for $j \in J$. That is, $x \in (A_j)^c$ for every $j \in J$ so by the definition of intersection, we conclude

$$x \in \bigcap_{j \in J} (A_j^c).$$

On the other hand, if

$$x \in \bigcap_{j \in J} (A_j^c),$$

then $x \in A_j^c$ for every $j \in J$ which by the definition of complement means that $x \notin A_j$ for any $j \in J$. But, by the definition of union, this is exactly the same as saying that

$$x \notin \bigcup_{j \in J} A_j.$$

In other words,

$$x \in \left(\bigcup_{j \in J} A_j \right)^c$$

and the proof is complete.

6. I am just going to give the answers. You should still prove that the two sets are equal.

(a) $\bigcup_{B \in \mathcal{B}} B = [1,2]$ and $\bigcap_{B \in \mathcal{B}} B = \{1\}$.

(b) $\bigcup_{B \in \mathcal{B}} B = (1,2)$ and $\bigcap_{B \in \mathcal{B}} B = \emptyset$. Note that $(1,1) = \emptyset$.

(c) $\bigcup_{B \in \mathcal{B}} B = [2,\infty)$ and $\bigcap_{B \in \mathcal{B}} B = \{2\}$.

(d) $\bigcup_{B \in \mathcal{B}} B = [0,5)$ and $\bigcap_{B \in \mathcal{B}} B = [2,3]$.