Mathematics 261 Final Exam – December 14, 2011

This exam is worth 150 points.

This exam has 9 problems on 3 numbered pages.

You have 3 hours to complete this exam. Please read all instructions carefully, and check your answers. Use the exam booklet for all of your written work, but not for your computer work. Indicate just your final answers on the answer sheet. You do not need to submit any computer code. For problems with multiple parts, all parts are equally weighted.

This exam is open book. You are permitted to use your class notes and textbook.

Instructor: Michael Kozdron

1. (*54 points*) Consider the function $f(x) = \sqrt{1+x^3}$ defined for $0 \le x \le 1$. This function does not have a simple antiderivative and so it must be analyzed numerically.

(a) Numerically determine a left-hand Riemann sum approximation for $\displaystyle\int_0^1 \sqrt{1+x^3}\,\mathrm{d}x$ using $n = 5000$ subintervals.

(b) Numerically determine a right-hand Riemann sum approximation for $\displaystyle\int_0^1 \sqrt{1+x^3}\,\mathrm{d}x$ using $n = 5000$ subintervals.

(c) Sketch a graph of $f(x) = \sqrt{1+x^3}$ for $0 \le x \le 1$ and carefully explain why your right-hand Riemann sum approximation from **(b)** is an upper bound for $\displaystyle\int_0^1 \sqrt{1+x^3}\,\mathrm{d}x$ while your left-hand Riemann sum approximation from **(a)** is a lower bound for it.

(d) Determine the value of the trapezoidal approximation to $\displaystyle\int_0^1 \sqrt{1+x^3}\,\mathrm{d}x$ by averaging your answers to **(a)** and **(b)**.

(e) Find the second degree Taylor polynomial about the point $y_0 = 0$ for $g(y) = \sqrt{1+y}$. In other words, you will find explicit values of a_0, a_1, and a_2 such that

$$\sqrt{1+y} \approx a_0 + a_1 y + a_2 y^2.$$

(f) Notice that $f(x) = g(x^3)$. Therefore, substitute $y = x^3$ into your answer for **(e)** to determine a portion of the Taylor expansion for $\sqrt{1+x^3}$. Your answer will have three non-zero terms. Call your answer $p(x)$.

(g) Consider the approximation $p(x)$ to $\sqrt{1+x^3}$ that you found in **(f)**. Determine the value of

$$\int_0^1 p(x)\,\mathrm{d}x.$$

(h) Perform a Monte Carlo simulation using $n = 5000$ randomly selected points to approximate the value of $\displaystyle\int_0^1 \sqrt{1+x^3}\,\mathrm{d}x$. Seed your random number generator with your student number. You may want to again consider a graph of $f(x) = \sqrt{1+x^3}$ for $0 \le x \le 1$ to determine an appropriate window for your Monte Carlo simulation. (This Monte Carlo simulation is known as the rejection method.)

(i) You now have three approximations to the value of $\displaystyle\int_0^1 \sqrt{1+x^3}\,\mathrm{d}x$, namely your answers to **(d)**, **(g)**, and **(h)**. Which answer do you think is most accurate in the sense that it most closely approximates the true value of this integral? Explain.

2. (*12 points*) Suppose that $f(x) = e^{2x} - 4 - x$.

(a) Prove that $f(x)$ must have a root in the interval $[0, 1]$.

(b) Use the secant method to determine this root with a tolerance of 10^{-6}.

3. (*12 points*) **(a)** Determine the quadratic Lagrange polynomial interpolating the following data.

x	-1	0	1
y	0	4	0

(b) Using the quadratic polynomial you found in **(a)**, predict the value of y if $x = 0.5$.

4. (*8 points*) Use the files for Shamir's Secret Sharing to encipher the word `LAGUERRE` to be shared among 3 people. Seed the random number generator with your student number and mention if you are using OCTAVE or MATLAB to answer this problem.

5. (*10 points*) Consider the following MATLAB/OCTAVE program. (It does not matter whether you analyze this as an OCTAVE program or as a MATLAB program. The answer is the same.) Does this program produce an infinite loop? If so, explain why. If not, what can you say about the final value of u?

```
u=1;
while 1+u > 1
    u=u/2;
end
```

6. (*14 points*) Suppose that the function $u = u(x, t)$ satisfies the one-dimensional heat equation

$$u_t = u_{xx}$$

for $t \geq 0$ and $-\pi \leq x \leq \pi$. Suppose further that u satisfies the boundary conditions

$$u(-\pi, t) = u(\pi, t) = 0$$

for $t > 0$, as well as the initial condition

$$u(x, 0) = \frac{1}{3}\sin(2x)$$

for $-\pi \leq x \leq \pi$. Use the method of separation of variables to determine the function u.

7. (*12 points*) Consider the function $f(x)$ defined for $-\pi \le x \le \pi$ by

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x \le \pi, \\ -1, & \text{if } -\pi \le x < 0. \end{cases}$$

Determine the Fourier series for f. (Note: The function f is known as a square wave and has important applications in electronics and signal processing.)

8. (*14 points*) Consider the set of integer lattice points in the 5×5 grid about the origin as a discretization of a square plate. Suppose that we assign value 1 to all sixteen exterior boundary points and boundary value 0 to the two horizontal neighbours of the origin as shown in the picture below. Suppose that the function $u(x, y)$ satisfies the discrete Dirichlet problem for steady state heat flow subject to the given boundary values. Determine exactly the values of $u(x, y)$ at all points labelled $(*)$ by setting up and solving a system of linear equations. You should note that by symmetry you just need to consider a system of three equations in three unknowns.

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & * & * & * & 1 \\ 1 & 0 & * & 0 & 1 \\ 1 & * & * & * & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array}$$

Note that the function $u(x, y)$ also equals the probability that a simple random walk starting at (x, y) visits a boundary point with value 1 before it visits a boundary point with value 0.

9. (*14 points*) Consider the set of integer lattice points in a 7×13 grid as a discretization of a rectangular plate. Suppose that we assign value 1 to all exterior boundary points and boundary value 0 to the central point as shown in the picture below. Suppose that the function $u(x, y)$ satisfies the discrete Dirichlet problem for steady state heat flow subject to the given boundary values. Use the method of relaxations with 400 iterations to approximate the value of $u(x, y)$ at all points labelled $(*)$.

$$\begin{array}{ccccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & * & * & * & * & * & * & * & * & * & * & * & 1 \\ 1 & * & * & * & * & * & * & * & * & * & * & * & 1 \\ 1 & * & * & * & * & * & 0 & * & * & * & * & * & 1 \\ 1 & * & * & * & * & * & * & * & * & * & * & * & 1 \\ 1 & * & * & * & * & * & * & * & * & * & * & * & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

You do not need to report all of the values that the method of relaxations produces. Instead you need to report the two values as indicated on your solutions page.

Answers to Math 261 Final Exam

Name: ______________________________ Circle Software Used: MATLAB or OCTAVE

1. (a)

1. (b)

1. (c)

1. (d)

1. (e)

1. (f)

1. (g)

1. (h)

1. (i)

2. (a)

2. (b)

3. (a)

3. (b)

4.

5.

6.

7.

8.

1	1	1	1	1
1	$\boxed{}$	$\boxed{}$	$\boxed{}$	1
1	0	$\boxed{}$	0	1
1	$\boxed{}$	$\boxed{}$	$\boxed{}$	1
1	1	1	1	1

9.

1	1	1	1	1	1	1	1	1	1	1	1	1
1	*	*	*	*	*	*	*	*	*	*	*	1
1	*	*	*	*	*	$\boxed{\text{A}}$	*	*	*	*	*	1
1	$\boxed{\text{B}}$	*	*	*	*	0	*	*	*	*	*	1
1	*	*	*	*	*	*	*	*	*	*	*	1
1	*	*	*	*	*	*	*	*	*	*	*	1
1	1	1	1	1	1	1	1	1	1	1	1	1

The boxes represent the positions for which you must give the value of $u(x, y)$ after performing the method of relaxations with 400 iterations. Enter those two numbers below.

A.

B.

Solutions to Math 261 Final Exam

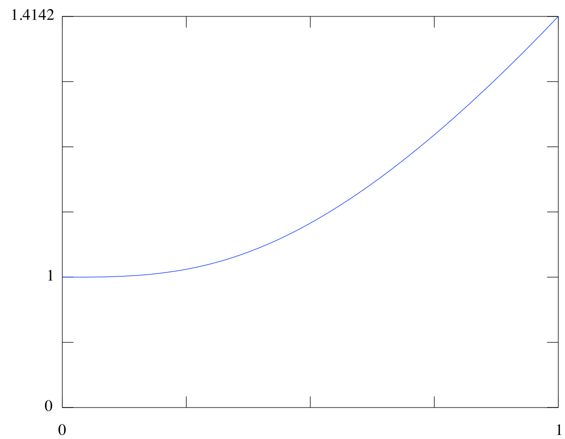
1. (a) Using OCTAVE, we find the left-hand Riemann sum approximation based on 5000 subintervals to be

```
> n = 5000;
> x = [0:1/n:(n-1)/n];
> LHS = sum(sqrt(1+x.^3)/n)
LHS =  1.11140655271187.
```

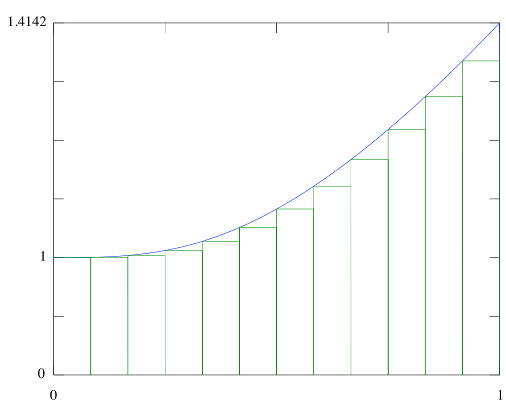
1. (b) Using OCTAVE, we find the right-hand Riemann sum approximation based on 5000 subintervals to be

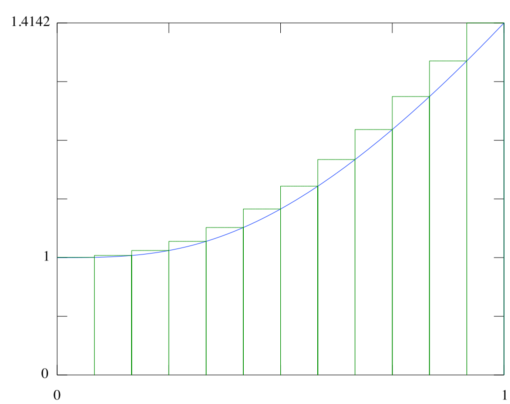
```
> n = 5000;
> y = [1/n:1/n:1];
> RHS = sum(sqrt(1+y.^3)/n)
RHS =  1.11148939542435.
```

1. (c) The graph of $f(x) = \sqrt{1 + x^3}$ is shown below.



Observe that f is strictly increasing for $0 \leq x \leq 1$. This implies that the left hand Riemann sum will be an under estimate of the true value of the integral (i.e., the true area under the curve), while the right hand Riemann sum will be an over estimate of the true value. That is, the LHS will be a lower bound and the RHS will be an upper bound.





1. (d) Using OCTAVE, we find the trapezoidal approximation based on 5000 subintervals to be

```
 > TRAP = (LHS + RHS)/2
TRAP =  1.11144797406811.
```

1. (e) If $g(y) = \sqrt{1+y}$, then

$$g'(y) = \frac{1}{2}(1+y)^{-1/2} \quad \text{and} \quad g''(y) = -\frac{1}{4}(1+y)^{-3/2}$$

so that $g(0) = 1$, $g'(0) = \frac{1}{2}$, and $g''(0) = -\frac{1}{4}$. Therefore, the second degree Taylor polynomial about the point $y = 0$ for $g(y) = \sqrt{1+y}$ is

$$g(0) + g'(0)y + \frac{g''(0)}{2}y^2 = 1 + \frac{y}{2} - \frac{y^2}{8}.$$

1. (f) Since $f(x) = g(x^3)$, the first three nonzero terms of the Taylor expansion of $\sqrt{1+x^3}$ are

$$p(x) = 1 + \frac{x^3}{2} - \frac{x^6}{8}.$$

1. (g) We find

$$\int_0^1 p(x)\,\mathrm{d}x = \int_0^1 1 + \frac{x^3}{2} - \frac{x^6}{8}\,\mathrm{d}x = 1 + \frac{1}{8} - \frac{1}{56} = \frac{62}{56} \doteq 1.10714285714286.$$

1. (h) The following program will perform the necessary Monte Carlo simulation. Notice that $f(1) = \sqrt{2}$. This means that we need to consider the window $0 \le x \le 1$ by $0 \le y \le \sqrt{2}$ which has an area of $\sqrt{2}$.

```
> rand('seed',123456789)
> x = rand(1,n); % values from 0 to 1
> y = sqrt(2)*rand(1,n); % values from 0 to sqrt(2)
> area = 1*sqrt(2);
> K = (y <= sqrt(1+x.^3));
> int = area*sum(K)/length(K)
int =  1.10563216306329
```

1. (i) The key to answering this problem is the observation from **(c)** that the left-hand and right-hand Riemann sums give rigorous bounds for true value of the integral; that is,

$$1.11140655271187 < \int_0^1 \sqrt{1+x^3}\,\mathrm{d}x < 1.11148939542435.$$

Notice that the answers to both **(g)** and **(h)** are *not* in this interval. The answer to **(d)** *is* in this interval and is therefore the closest approximation to the true value from among **(d)**, **(g)**, and **(h)**.

2. **(a)** Since f is continuous on $[0,1]$ and satisfies $f(0) = 1 - 4 - 0 = -3 < 0$ and $f(1) = e^2 - 4 - 1 \doteq 2.389 > 0$, we conclude from the Intermediate Value Theorem that f must have a root in $[0,1]$.

2. **(b)** Using OCTAVE with the secant method algorithm from Assignment #4 gives

```
> secant(0,1,1E-6,1000)
ans =  0.782479575861968.
```

3. **(a)** From the data given, our nodes are $[x_0, x_1, x_2] = [-1, 0, 1]$ and $[y_0, y_1, y_2] = [0, 4, 0]$. Therefore,

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 0)(x - 1)}{(-1 - 0)(-1 - 1)} = \frac{x^2 - x}{2},$$
$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x + 1)(x - 1)}{(0 + 1)(0 - 1)} = 1 - x^2,$$
$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x + 1)(x - 0)}{(1 + 1)(1 - 0)} = \frac{x^2 + x}{2},$$

and so

$$\begin{aligned} P_2(x) &= L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2 \\ &= \left(\frac{x^2 - x}{2}\right) \cdot 0 + (1 - x^2) \cdot 4 + \left(\frac{x^2 + x}{2}\right) \cdot 0 \\ &= 4(1 - x^2). \end{aligned}$$

3. **(b)** Since $P_2(x) = 4(1 - x^2)$, our predicted value of y if $x = 0.5$ is

$$P_2(0.5) = 4(1 - (0.5)^2) = 3.$$

4. Using OCTAVE, we find that enciphering the word `LAGUERRE` to be shared among three people using the files for Shamir's Secret Sharing produces the following.

```
> rand('seed',123456789)
> secret('LAGUERRE',3)
ans =  88430499887   88430578788   88430658657
```

5. Notice that this program takes the initial value of $u = 1$ and successively halves it. Mathematically, this is an infinite process and produces the following sequence:

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots.$$

Thus, if u is any element in the sequence, then u is necessarily strictly greater than 0 implying that $1 + u$ is necessarily greater than 1. However, a computer program's built-in numerical accuracy is limited and so the program has a largest positive number (known as the machine epsilon) which can be added to 1 to produce the answer 1. This is the reason that the given MATLAB/OCTAVE program does *not* produce an infinite loop. Executing this program in OCTAVE gives the value `u = 1.11022302462516e-16`. Note that if you enter $1 + \mathrm{u}$ with this value of `u` in OCTAVE, the answer is 1. (That is, `u` is the largest positive number such that $1 + \mathrm{u} = 1$.

6. Assume that $u(x,t) = X(x)T(t)$ for some function X of x only and some function T of t only. If $u_t = u_{xx}$, then

$$X(x)T'(t) = X''(x)T(t) \quad \text{or, equivalently,} \quad \frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}.$$

The only way that a function of x only can equal a function of t only for all values of $x \in [-\pi, \pi]$ and $t \geq 0$ is if those functions are equal to a constant, say λ. Hence,

$$\frac{X''(x)}{X(x)} = \lambda \quad \text{and} \quad \frac{T'(t)}{T(t)} = \lambda.$$

The first equation implies that $X''(x) - \lambda X(x) = 0$ which has arbitrary solutions

$$X(x) = C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x).$$

The second equation implies that $T'(t) - \lambda T(t) = 0$ which has arbitrary solutions

$$T(t) = C_3 e^{\lambda t}.$$

Hence,

$$u(x,t) = X(x)T(t) = K_1 e^{\lambda t} \sin(\sqrt{\lambda}x) + K_2 e^{\lambda t} \cos(\sqrt{\lambda}x)$$

where K_1 and K_2 are arbitrary constants. Observe that

$$u(x,0) = X(x)T(0) = K_1 e^{\lambda 0} \sin(\sqrt{\lambda}x) + K_2 e^{\lambda 0} \cos(\sqrt{\lambda}x) = K_1 \sin(\sqrt{\lambda}x) + K_2 \cos(\sqrt{\lambda}x)$$

so that the initial condition

$$u(x,0) = \frac{1}{3}\sin(2x)$$

implies that

$$\frac{1}{3}\sin(2x) = K_1 \sin(\sqrt{\lambda}x) + K_2 \cos(\sqrt{\lambda}x).$$

This forces $K_1 = \frac{1}{3}$, $\sqrt{\lambda} = 2$, and $K_2 = 0$ so that

$$u(x,t) = \frac{1}{3}e^{4t}\sin(2x).$$

Observe that

$$u(-\pi,t) = \frac{1}{3}e^{4t}\sin(-2\pi) = 0$$

and

$$u(\pi,t) = \frac{1}{3}e^{4t}\sin(2\pi) = 0$$

as required.

7. The Fourier series for f is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x.$$

Now

$$\begin{aligned}
a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x &= \frac{1}{\pi}\int_{0}^{\pi} \cos(nx)\,\mathrm{d}x - \frac{1}{\pi}\int_{-\pi}^{0} \cos(nx)\,\mathrm{d}x \\
&= \frac{1}{n\pi}\sin(nx)\Big|_{x=0}^{x=\pi} - \frac{1}{n\pi}\sin(nx)\Big|_{x=-\pi}^{x=0} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x &= \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)\,\mathrm{d}x - \frac{1}{\pi}\int_{-\pi}^{0} \sin(nx)\,\mathrm{d}x \\
&= -\frac{1}{n\pi}\cos(nx)\Big|_{x=0}^{x=\pi} + \frac{1}{n\pi}\cos(nx)\Big|_{x=-\pi}^{x=0} \\
&= -\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi}\cos(0) + \frac{1}{n\pi}\cos(0) - \frac{1}{n\pi}\cos(-n\pi) \\
&= \frac{2}{n\pi} - \frac{2}{n\pi}\cos(n\pi) \\
&= \frac{2}{n\pi}(1-\cos(n\pi)) \\
&= \frac{2}{n\pi}(1-(-1)^n)
\end{aligned}$$

which implies that

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n}\sin(nx) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin((2k-1)x).$$

8. By symmetry, there are three possible values of $u(x,y)$ labelled a, b, and c as shown.

$$\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
1 & c & b & c & 1 \\
1 & 0 & a & 0 & 1 \\
1 & c & b & c & 1 \\
1 & 1 & 1 & 1 & 1
\end{array}$$

The value of $u(x,y)$ is equal to the average of the values at its neighbours; that is,

$$u(x,y) = \frac{1}{4}\left[u(x+1,y) + u(x-1,y) + u(x,y+1) + u(x,y-1)\right].$$

Therefore, a, b, and c satisfy the system of equations

$$\begin{cases} a = \dfrac{1}{4}[b+b+0+0] \\ b = \dfrac{1}{4}[a+c+c+1] \\ c = \dfrac{1}{4}[1+1+0+b] \end{cases} .$$

Simplifying gives

$$2a = b, \quad 4b = 2c + a + 1, \quad 4c = 2 + b.$$

Substituting the first and third equations into the second equation gives

$$8b = 4c + 2a + 2 = (2+b) + b + 2 \quad \text{or, equivalently,} \quad b = \frac{2}{3}.$$

From the first equation

$$a = \frac{1}{2}b = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

and from the third equation

$$c = \frac{1}{2} + \frac{1}{4}b = \frac{1}{2} + \frac{1}{4} \cdot \frac{2}{3} = \frac{2}{3}.$$

Thus, the final answer is shown.

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 2/3 & 2/3 & 2/3 & 1 \\ 1 & 0 & 1/3 & 0 & 1 \\ 1 & 2/3 & 2/3 & 2/3 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array}$$

9. The following program will perform the method of relaxations on an $N \times M$ matrix with value 1 at all exterior boundary points and boundary value 0 at the central point.

```
function A = relax(N,M,K)
% create an NxM matrix with 1 on the exterior boundary and 0 in the centre.
% N and M must be odd
% relax the interior entries in a systematic way
% K is the number of iterations

if mod(N,2)==0 || mod(M,2)==0
    error('The number of rows and columns must be odd')
end

C1 = ceil(N/2);
C2 = ceil(M/2);
```

```
A = zeros(N,M);
A(1,:) = 1;
A(N,:) = 1;
A(:,1) = 1;
A(:,M) = 1;

for k=1:K
    for i=2:N-1
        for j=2:M-1
            if (i~=C1) || (j~=C2)    % Ensure not on inner boundary
                A(i,j)=0.25*(A(i,j-1)+A(i,j+1)+A(i-1,j)+A(i+1,j));
            end
        end
    end
end
```

Executing the program `relax(7,13,400)` produces the required output. The values of A and B are

$$A = 0.541127567623125 \text{ and } B = 0.966836356353507.$$