Math 261 Fall 2011
Solutions to Assignment #5

1. The following commands load the data and create separate vectors for `x` and `y`.

```
load data.m
x=data(:,1)
y=data(:,2)
```

Note that if you are using MATLAB, you want to `load data.mat` instead. Now define `xbar` and `ybar` to be the mean of `x` and `y`, respectively, and compute `m` and `b`.

```
xbar = sum(x) / length(x)
ybar = sum(y) / length(y)
m = sum((x-xbar).*(y-ybar)) / sum((x-xbar).*(x-xbar))
b = ybar - m*xbar
```

The answers are

```
m =  0.471146705905806
b = -3.25490196668521
```

so that the equation of the least-squares line (i.e., regression line or line of best fit) is

$$y = 0.471146705905806x - 3.25490196668521.$$

2. Here are two files that you could use in order to solve this problem. The first one was written by Sarah and implements the pseudocode distributed in class. It is called `lg.m`.

```
function Pz = lg(x,y,z)
% Compute Lagrange polynomial P(x) and evaluate at z
%inputs
%   x - a vector of points
%   y - a vector of f(x)
%   z - a point to be evalutated
%output Pz = P(z)

%Initialize Variables
Pz=0;
n=length(x);
L=ones(n,1);

for i=1:n
    for j=1:n
        if i ~= j
            L(i) = (z-x(j))/(x(i)-x(j))*L(i);
        end
    end
    Pz = L(i)*y(i)+Pz;
end
end
```

Here is a second solution that I wrote and is called `lagrange.m`.

```
function Pz=lagrange(x,y,z)
% Evaluates the nth Lagrange interpolating polynomial P
%     to the function f using (n+1) nodes
% Input x: row vector containing (n+1) nodes
% Input y: row vector containing function values at the nodes
% Input z: point you are interested in approximating the function at
% Output: value of P(z),

Pz=0;
n=length(x);
L=ones(1,n);

L(1)=prod(z-[x(2:n)])/(prod(x(1)-[x(2:n)]));

L(n)=prod(z-[x(1:n-1)])/(prod(x(n)-[x(1:n-1)]));

for i = [2:n-1]
L(i)=prod(z-[x(1:i-1)]).*prod(z-[x(i+1:n)])/(prod(x(i)-[x(1:i-1)]).*prod(x(i)-[x(i+1:n)]));
end

Pz=sum(L.*y);
end
```

Implementing `lagrange.m` gives

```
x = [1 2 3];
y = [149674925 386437459 729429125];
z = 0;
lagrange(x,y,z)
ans =  19141523
```

If we now convert `19141523` to `19 14 15 23` and then convert the numbers to letters, we find the secret word is `SNOW`.

3. **(a)** (Exercise 6(a) on page 115) In order to approximate $f(0.43)$ using the Lagrange interpolating polynomials of degrees 1, 2, and 3, we require 2, 3, and 4 nodes, respectively. We will use the nodes closest to 0.43 for the approximations. Hence, let

```
x = [0 0.25 0.5 0.75];
y = [1 1.64872 2.71828 4.48169];
z = 0.43;
```

so that the degree 1 approximation is

```
lagrange(x(2:3),y(2:3),z)
ans = 2.41880320000000,
```

(continued)

the degree 2 approximation is

```
lagrange(x(2:4),y(2:4),z)
ans = 2.34886312000000,
```

and the degree 3 approximation is

```
lagrange(x,y,z)
ans = 2.36060473408000.
```

If you used the first 3 nodes for the degree 2 approximation instead, then you would find

```
lagrange(x(1:3),y(1:3),z)
ans = 2.37638252800000.
```

3. (b) (Exercise 18 on page 116) Using `lagrange.m` with

```
x = [1950 1960 1970 1980 1990 2000];
y = [151326 179323 203302 226542 249633 281422];
```

gives our approximation for the population in 1940 as

```
lagrange(x,y,1940)
ans = 102397,
```

our approximation for the population in 1975 as

```
lagrange(x,y,1975)
ans = 215042.75,
```

and our approximation for the population in 2020 as

```
lagrange(x,y,2020)
ans = 513443.
```

Our 1975 figure is probably quite accurate since it is an interior data point, while our 2020 is probably not very accurate since this prediction assumes that the rate of growth of the population is constant.

4. (Exercise 10 on page 115) If $f(x) = \sqrt{x - x^2}$ then our nodes are $[x_0, x_1, x_2] = [0, x_1, 1]$ and $[y_0, y_1, y_2] = [f(x_0), f(x_1), f(x_2)] = [f(0), f(x_1), f(1)] = [0, \sqrt{x_1 - x_1^2}, 0]$. Therefore,

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - x_1)(x - 1)}{x_1},$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{x(x - 1)}{x_1(x_1 - 1)},$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{x(x - x_1)}{(1 - x_1)},$$

and so

$$P_2(x) = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2 = \frac{(x-x_1)(x-1)}{x_1} \cdot 0 + \frac{x(x-1)}{x_1(x_1-1)} \cdot y_1 + \frac{x(x-x_1)}{(1-x_1)} \cdot 0$$
$$= \frac{x(x-1)}{x_1(x_1-1)} \cdot \sqrt{x_1 - x_1^2}$$
$$= -\frac{x(x-1)}{\sqrt{x_1(1-x_1)}}.$$

If we now consider $f(x) - P_2(x)$, then

$$f(x) - P_2(x) = \sqrt{x - x^2} + \frac{x(x-1)}{\sqrt{x_1(1-x_1)}}.$$

Hence, $f(0.5) - P_2(0.5) = -0.25$ implies

$$\sqrt{(0.5) - (0.5)^2} + \frac{(0.5)(0.5-1)}{\sqrt{x_1(1-x_1)}} = -0.25$$

or, equivalently,

$$0.5 - \frac{0.25}{\sqrt{x_1(1-x_1)}} = -0.25$$

which implies

$$2 - \frac{1}{\sqrt{x_1(1-x_1)}} = -1 \quad \text{and so} \quad \sqrt{x_1(1-x_1)} = \frac{1}{3}.$$

Expanding the square gives

$$x_1^2 - x_1 = -\frac{1}{9}.$$

The two roots of this equation may be found by completing the square, namely

$$x_1^2 - x_1 + \frac{1}{4} = -\frac{1}{9} + \frac{1}{4} \quad \text{or} \quad \left(x_1 - \frac{1}{2}\right)^2 = \frac{5}{36}$$

and so

$$x_1 = \frac{1}{2} - \sqrt{\frac{5}{36}} \quad \text{or} \quad x_1 = \frac{1}{2} + \sqrt{\frac{5}{36}}.$$

The largest of these is therefore

$$x_1 = \frac{1}{2} + \sqrt{\frac{5}{36}} = \frac{3+\sqrt{5}}{6} \approx 0.872677996249965.$$