Math 261 Fall 2011
Solutions to Assignment #1

Suppose that $f(x) = \sqrt{1-x^2}$ so that $\pi = 4\int_0^1 f(x)\,\mathrm{d}x$. Thus, in order to approximate $\pi$, we will approximate

$$\int_0^1 f(x)\,\mathrm{d}x \qquad (*)$$

and multiply by 4.

**(a)** Using $n$ partition points, we find the left-hand Riemann sum (LHS) approximation to $(*)$ is

$$\frac{1}{n}\sum_{j=0}^{n-1} f(\tfrac{j}{n}) = \frac{1}{n}\sum_{j=0}^{n-1}\sqrt{1-\frac{j^2}{n^2}} = \frac{1}{n^2}\sum_{j=0}^{n-1}\sqrt{n^2-j^2}.$$

**(b)** Using $n$ partition points, we find the left-hand Riemann sum (LHS) approximation to $(*)$ is

$$\frac{1}{n}\sum_{j=1}^{n} f(\tfrac{j}{n}) = \frac{1}{n}\sum_{j=1}^{n}\sqrt{1-\frac{j^2}{n^2}} = \frac{1}{n^2}\sum_{j=1}^{n}\sqrt{n^2-j^2}.$$

**(c)** When $n = 3$, the value of the LHS is

$$\frac{1}{3^2}\left[\sqrt{3^2-0^2}+\sqrt{3^2-1^2}+\sqrt{3^2-2^2}\right] = \frac{\sqrt{9}+\sqrt{8}+\sqrt{5}}{9} \approx 0.896055$$

so that an approximation to $\pi$ is

$$4 \times \frac{\sqrt{9}+\sqrt{8}+\sqrt{5}}{9} \approx 3.584220.$$

When $n = 6$, the value of the LHS is

$$\begin{aligned}&\frac{1}{6^2}\left[\sqrt{6^2-0^2}+\sqrt{6^2-1^2}+\sqrt{6^2-2^2}+\sqrt{6^2-3^2}+\sqrt{6^2-4^2}+\sqrt{6^2-5^2}\right]\\ &= \frac{\sqrt{36}+\sqrt{35}+\sqrt{32}+\sqrt{27}+\sqrt{20}+\sqrt{11}}{36}\\ &\approx 0.848829\end{aligned}$$

so that an approximation to $\pi$ is

$$4 \times \frac{\sqrt{36}+\sqrt{35}+\sqrt{32}+\sqrt{27}+\sqrt{20}+\sqrt{11}}{36} \approx 3.395316.$$

When $n = 3$, the value of the RHS is

$$\frac{1}{3^2}\left[\sqrt{3^2-1^2}+\sqrt{3^2-2^2}+\sqrt{3^2-3^2}\right] = \frac{\sqrt{8}+\sqrt{5}+\sqrt{0}}{9} \approx 0.562722$$

so that an approximation to $\pi$ is

$$4 \times \frac{\sqrt{8}+\sqrt{5}}{9} \approx 2.250887.$$

When $n = 6$, the value of the LHS is

$$\begin{aligned}
&\frac{1}{6^2}\left[\sqrt{6^2-1^2}+\sqrt{6^2-2^2}+\sqrt{6^2-3^2}+\sqrt{6^2-4^2}+\sqrt{6^2-5^2}+\sqrt{6^2-6^2}\right] \\
&= \frac{\sqrt{35}+\sqrt{32}+\sqrt{27}+\sqrt{20}+\sqrt{11}+\sqrt{0}}{36} \\
&\approx 0.682162
\end{aligned}$$

so that an approximation to $\pi$ is

$$4 \times \frac{\sqrt{35}+\sqrt{32}+\sqrt{27}+\sqrt{20}+\sqrt{11}}{36} \approx 2.728650.$$

**(d)** Using $n$ partition points, we find the midpoint Riemann sum (MPS) approximation to $(*)$ is

$$\frac{1}{n}\sum_{j=0}^{n-1} f\left(\tfrac{j}{n}+\tfrac{1}{2n}\right) = \frac{1}{n}\sum_{j=0}^{n-1} f\left(\tfrac{2j+1}{2n}\right) = \frac{1}{n}\sum_{j=0}^{n-1}\sqrt{1-\frac{(2j+1)^2}{(2n)^2}} = \frac{1}{2n^2}\sum_{j=0}^{n-1}\sqrt{4n^2-(2j+1)^2}.$$

**(e)** When $n = 3$, the value of the MPS is

$$\begin{aligned}
&\frac{1}{2\cdot 3^2}\left[\sqrt{4\cdot 3^2-1^2}+\sqrt{4\cdot 3^2-3^2}+\sqrt{4\cdot 3^2-5^2}\right] \\
&= \frac{\sqrt{35}+\sqrt{27}+\sqrt{11}}{18} \\
&\approx 0.801603
\end{aligned}$$

so that an approximation to $\pi$ is

$$4 \times \frac{\sqrt{35}+\sqrt{27}+\sqrt{11}}{18} \approx 3.206413.$$

When $n = 6$, the value of the MPS is

$$\begin{aligned}
&\frac{1}{2\cdot 6^2}\left[\sqrt{4\cdot 6^2-1^2}+\sqrt{4\cdot 6^2-3^2}+\sqrt{4\cdot 6^2-5^2}+\sqrt{4\cdot 6^2-7^2}+\sqrt{4\cdot 6^2-9^2}+\sqrt{4\cdot 6^2-11^2}\right] \\
&= \frac{\sqrt{143}+\sqrt{135}+\sqrt{119}+\sqrt{95}+\sqrt{63}+\sqrt{23}}{72} \\
&\approx 0.791192
\end{aligned}$$

so that an approximation to $\pi$ is

$$4 \times \frac{\sqrt{143}+\sqrt{135}+\sqrt{119}+\sqrt{95}+\sqrt{63}+\sqrt{23}}{72} \approx 3.164767.$$

**(f)** If $f(x) = \sqrt{1+x}$, then taking successive derivatives gives

$$\begin{aligned}
f'(x) &= \frac{1}{2}(1+x)^{-1/2}, \\
f''(x) &= -\frac{1}{2\cdot 2}(1+x)^{-3/2},
\end{aligned}$$

$$f'''(x) = \frac{3}{2\cdot 2\cdot 2}(1+x)^{-5/2},$$
$$f^{(4)}(x) = -\frac{3\cdot 5}{2\cdot 2\cdot 2\cdot 2}(1+x)^{-7/2},$$

so that the pattern (for $j = 2, 3, \ldots$) is

$$f^{(j)}(x) = (-1)^{j-1}\frac{1\cdot 3\cdots(2(j-1)-1)}{2^j}(1+x)^{-(2j-1)}.$$

Notice that for $j \geq 2$, the numerator of the $j$th derivative is the product of the first $j-1$ odd numbers. Also notice that we can write a product of odd numbers as a product of numbers divided by a product of even numbers; that is,

$$1\cdot 3\cdots(2(j-1)-1) = \frac{1\cdot 2\cdot 3\cdot 4\cdots(2(j-1)-1)}{2\cdot 4\cdots[2(j-1)-2]} = \frac{(2(j-1)-1)!}{2^{j-2}[1\cdot 2\cdots(j-2)]} = \frac{(2j-3)!}{2^{j-2}(j-2)!}.$$

and so for $j = 2, 3, \ldots$ we have

$$f^{(j)}(x) = (-1)^{j-1}\frac{(2j-3)!}{2^{2j-2}(j-2)!}(1+x)^{-(2j-1)}.$$

Hence, the Taylor series for $\sqrt{1+x}$ about the point $x = 0$ is

$$\sqrt{1+x} = \sum_{j=0}^{\infty}\frac{f^{(j)}(0)}{j!}x^j = 1 + \frac{1}{2}x + \sum_{j=2}^{\infty}(-1)^{j-1}\frac{(2j-3)!}{2^{2j-2}(j-2)!j!}x^j.$$

Furthermore, the Taylor series about the point $a = 0$ for $\sqrt{1-x^2}$ is

$$\begin{aligned}\sqrt{1-x^2} &= 1 + \frac{1}{2}(-x^2) + \sum_{j=2}^{\infty}(-1)^{j-1}\frac{(2j-3)!}{2^{2j-2}(j-2)!j!}(-x^2)^j \\ &= 1 - \frac{x^2}{2} - \sum_{j=2}^{\infty}\frac{(2j-3)!}{2^{2j-2}(j-2)!j!}x^{2j}.\end{aligned}$$

Note that $(-1)^{j-1}(-1)^j = (-1)^{2j-1} = -1$ for any $j = 2, 3, \ldots$.

**(g)** Written out explicitly, the first five nonzero terms in the Taylor series expansion about the point $a = 0$ for $\sqrt{1-x^2}$ are

$$1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128}$$

and so

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x &\approx \int_0^1\left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128}\right)\mathrm{d}x \\ &= 1 - \frac{1}{2\cdot 3} - \frac{1}{8\cdot 5} - \frac{1}{16\cdot 7} - \frac{5}{128\cdot 9} \\ &= 1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152} \\ &= \frac{32057}{40320} \approx 0.795064.\end{aligned}$$

This implies that an approximation for $\pi$ is

$$\pi \approx 4\cdot\frac{32057}{40320} = \frac{32057}{10080} \approx 3.180258.$$