

CS 261 Fall 2011
Solutions to Assignment #1

Suppose that $f(x) = \sqrt{1-x^2}$ so that $\pi = 4 \int_0^1 f(x) dx$. Thus, in order to approximate π , we will approximate

$$\int_0^1 f(x) dx \quad (*)$$

and multiply by 4.

(a) Using n partition points, we find the left-hand Riemann sum (LHS) approximation to (*) is

$$\frac{1}{n} \sum_{j=0}^{n-1} f\left(\frac{j}{n}\right) = \frac{1}{n} \sum_{j=0}^{n-1} \sqrt{1 - \frac{j^2}{n^2}} = \frac{1}{n^2} \sum_{j=0}^{n-1} \sqrt{n^2 - j^2}.$$

(b) Using n partition points, we find the left-hand Riemann sum (LHS) approximation to (*) is

$$\frac{1}{n} \sum_{j=1}^n f\left(\frac{j}{n}\right) = \frac{1}{n} \sum_{j=1}^n \sqrt{1 - \frac{j^2}{n^2}} = \frac{1}{n^2} \sum_{j=1}^n \sqrt{n^2 - j^2}.$$

(c) When $n = 3$, the value of the LHS is

$$\frac{1}{3^2} \left[\sqrt{3^2 - 0^2} + \sqrt{3^2 - 1^2} + \sqrt{3^2 - 2^2} \right] = \frac{\sqrt{9} + \sqrt{8} + \sqrt{5}}{9} \approx 0.896055$$

so that an approximation to π is

$$4 \times \frac{\sqrt{9} + \sqrt{8} + \sqrt{5}}{9} \approx 3.584220.$$

When $n = 6$, the value of the LHS is

$$\begin{aligned} & \frac{1}{6^2} \left[\sqrt{6^2 - 0^2} + \sqrt{6^2 - 1^2} + \sqrt{6^2 - 2^2} + \sqrt{6^2 - 3^2} + \sqrt{6^2 - 4^2} + \sqrt{6^2 - 5^2} \right] \\ &= \frac{\sqrt{36} + \sqrt{35} + \sqrt{32} + \sqrt{27} + \sqrt{20} + \sqrt{11}}{36} \\ &\approx 0.848829 \end{aligned}$$

so that an approximation to π is

$$4 \times \frac{\sqrt{36} + \sqrt{35} + \sqrt{32} + \sqrt{27} + \sqrt{20} + \sqrt{11}}{36} \approx 3.395316.$$

When $n = 3$, the value of the RHS is

$$\frac{1}{3^2} \left[\sqrt{3^2 - 1^2} + \sqrt{3^2 - 2^2} + \sqrt{3^2 - 3^2} \right] = \frac{\sqrt{8} + \sqrt{5} + \sqrt{0}}{9} \approx 0.562722$$

so that an approximation to π is

$$4 \times \frac{\sqrt{8} + \sqrt{5}}{9} \approx 2.250887.$$

When $n = 6$, the value of the LHS is

$$\begin{aligned} & \frac{1}{6^2} \left[\sqrt{6^2 - 1^2} + \sqrt{6^2 - 2^2} + \sqrt{6^2 - 3^2} + \sqrt{6^2 - 4^2} + \sqrt{6^2 - 5^2} + \sqrt{6^2 - 6^2} \right] \\ &= \frac{\sqrt{35} + \sqrt{32} + \sqrt{27} + \sqrt{20} + \sqrt{11} + \sqrt{0}}{36} \\ &\approx 0.682162 \end{aligned}$$

so that an approximation to π is

$$4 \times \frac{\sqrt{35} + \sqrt{32} + \sqrt{27} + \sqrt{20} + \sqrt{11}}{36} \approx 2.728650.$$

(d) Using n partition points, we find the midpoint Riemann sum (MPS) approximation to (*) is

$$\frac{1}{n} \sum_{j=0}^{n-1} f\left(\frac{j}{n} + \frac{1}{2n}\right) = \frac{1}{n} \sum_{j=0}^{n-1} f\left(\frac{2j+1}{2n}\right) = \frac{1}{n} \sum_{j=0}^{n-1} \sqrt{1 - \frac{(2j+1)^2}{(2n)^2}} = \frac{1}{2n^2} \sum_{j=0}^{n-1} \sqrt{4n^2 - (2j+1)^2}.$$

(e) When $n = 3$, the value of the MPS is

$$\begin{aligned} & \frac{1}{2 \cdot 3^2} \left[\sqrt{4 \cdot 3^2 - 1^2} + \sqrt{4 \cdot 3^2 - 3^2} + \sqrt{4 \cdot 3^2 - 5^2} \right] \\ &= \frac{\sqrt{35} + \sqrt{27} + \sqrt{11}}{18} \\ &\approx 0.801603 \end{aligned}$$

so that an approximation to π is

$$4 \times \frac{\sqrt{35} + \sqrt{27} + \sqrt{11}}{18} \approx 3.206413.$$

When $n = 6$, the value of the MPS is

$$\begin{aligned} & \frac{1}{2 \cdot 6^2} \left[\sqrt{4 \cdot 6^2 - 1^2} + \sqrt{4 \cdot 6^2 - 3^2} + \sqrt{4 \cdot 6^2 - 5^2} + \sqrt{4 \cdot 6^2 - 7^2} + \sqrt{4 \cdot 6^2 - 9^2} + \sqrt{4 \cdot 6^2 - 11^2} \right] \\ &= \frac{\sqrt{143} + \sqrt{135} + \sqrt{119} + \sqrt{95} + \sqrt{63} + \sqrt{23}}{72} \\ &\approx 0.791192 \end{aligned}$$

so that an approximation to π is

$$4 \times \frac{\sqrt{143} + \sqrt{135} + \sqrt{119} + \sqrt{95} + \sqrt{63} + \sqrt{23}}{72} \approx 3.164767.$$

(f) If $f(x) = \sqrt{1+x}$, then taking successive derivatives gives

$$\begin{aligned} f'(x) &= \frac{1}{2}(1+x)^{-1/2}, \\ f''(x) &= -\frac{1}{2 \cdot 2}(1+x)^{-3/2}, \end{aligned}$$

$$f'''(x) = \frac{3}{2 \cdot 2 \cdot 2}(1+x)^{-5/2},$$

$$f^{(4)}(x) = -\frac{3 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 2}(1+x)^{-7/2},$$

so that the pattern (for $j = 2, 3, \dots$) is

$$f^{(j)}(x) = (-1)^{j-1} \frac{1 \cdot 3 \cdots (2(j-1) - 1)}{2^j} (1+x)^{-(2j-1)}.$$

Notice that for $j \geq 2$, the numerator of the j th derivative is the product of the first $j-1$ odd numbers. Also notice that we can write a product of odd numbers as a product of numbers divided by a product of even numbers; that is,

$$1 \cdot 3 \cdots (2(j-1) - 1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2(j-1) - 1)}{2 \cdot 4 \cdots [2(j-1) - 2]} = \frac{(2(j-1) - 1)!}{2^{j-2}[1 \cdot 2 \cdots (j-2)]} = \frac{(2j-3)!}{2^{j-2}(j-2)!}.$$

and so for $j = 2, 3, \dots$ we have

$$f^{(j)}(x) = (-1)^{j-1} \frac{(2j-3)!}{2^{2j-2}(j-2)!} (1+x)^{-(2j-1)}.$$

Hence, the Taylor series for $\sqrt{1+x}$ about the point $x = 0$ is

$$\sqrt{1+x} = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j = 1 + \frac{1}{2}x + \sum_{j=2}^{\infty} (-1)^{j-1} \frac{(2j-3)!}{2^{2j-2}(j-2)!j!} x^j.$$

Furthermore, the Taylor series about the point $a = 0$ for $\sqrt{1-x^2}$ is

$$\begin{aligned} \sqrt{1-x^2} &= 1 + \frac{1}{2}(-x^2) + \sum_{j=2}^{\infty} (-1)^{j-1} \frac{(2j-3)!}{2^{2j-2}(j-2)!j!} (-x^2)^j \\ &= 1 - \frac{x^2}{2} - \sum_{j=2}^{\infty} \frac{(2j-3)!}{2^{2j-2}(j-2)!j!} x^{2j}. \end{aligned}$$

Note that $(-1)^{j-1}(-1)^j = (-1)^{2j-1} = -1$ for any $j = 2, 3, \dots$

- (g) Written out explicitly, the first five nonzero terms in the Taylor series expansion about the point $a = 0$ for $\sqrt{1-x^2}$ are

$$1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128}$$

and so

$$\begin{aligned} \int_0^1 \sqrt{1-x^2} dx &\approx \int_0^1 \left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128} \right) dx \\ &= 1 - \frac{1}{2 \cdot 3} - \frac{1}{8 \cdot 5} - \frac{1}{16 \cdot 7} - \frac{5}{128 \cdot 9} \\ &= 1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152} \\ &= \frac{32057}{40320} \approx 0.795064. \end{aligned}$$

This implies that an approximation for π is

$$\pi \approx 4 \cdot \frac{32057}{40320} = \frac{32057}{10080} \approx 3.180258.$$