Statistics 252–Mathematical Statistics
Winter 2016 (201610)
Final Exam Solutions

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1. (a) The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \prod_{i=1}^{n} \frac{3y_i^2}{\theta^3} \exp\left\{-\frac{y_i^3}{\theta^3}\right\} = 3^n \theta^{-3n} \left(\prod_{i=1}^{n} y_i\right)^2 \exp\left\{-\frac{1}{\theta^3}\sum_{i=1}^{n} y_i^3\right\}$$

provided $y_1 > 0, \dots, y_n > 0$, and $\theta > 0$.

1. (b) The log-likelihood function is

$$\ell(\theta) = n \log 3 - 3n \log \theta + 2\sum_{i=1}^{n} \log y_i - \frac{1}{\theta^3}\sum_{i=1}^{n} y_i^3.$$

Taking derivatives implies

$$\ell'(\theta) = -\frac{3n}{\theta} + \frac{3}{\theta^4}\sum_{i=1}^{n} y_i^3 \quad \text{and} \quad \ell''(\theta) = \frac{3n}{\theta^2} - \frac{12}{\theta^5}\sum_{i=1}^{n} y_i^3.$$

Setting $\ell(\theta_0) = 0$ implies the only critical point θ_0 satisfies

$$\frac{3n}{\theta_0} = \frac{3}{\theta_0^4}\sum_{i=1}^{n} y_i^3, \quad \text{or, equivalently,} \quad \theta_0 = \left(\frac{1}{n}\sum_{i=1}^{n} y_i^3\right)^{1/3}.$$

Since

$$\ell''(\theta_0) = \frac{3}{\theta_0^2}\left(n - \frac{4}{\theta_0^3}\sum_{i=1}^{n} y_i^3\right) = \frac{3}{\theta_0^2}(n - 4n) = -\frac{9n}{\theta_0^2} < 0$$

we conclude from the second derivative test that θ_0 is, indeed, a global minimum. Thus,

$$\hat{\theta}_{\text{MLE}} = \left(\frac{1}{n}\sum_{i=1}^{n} Y_i^3\right)^{1/3}$$

as required.

1. (c) Let

$$h(y_1, \dots, y_n) = 3^n \left(\prod_{i=1}^{n} y_i\right)^2, \quad u = \left(\frac{1}{n}\sum_{i=1}^{n} y_i^3\right)^{1/3}, \quad \text{and} \quad g(u, \theta) = \theta^{-3n} \exp\left\{-\frac{nu^3}{\theta^3}\right\}.$$

Since $L(\theta) = h(y_1, \dots, y_n)g(u, \theta)$, we conclude from the factorization theorem that $\hat{\theta}_{\text{MLE}}$ is, in fact, sufficient for the estimation of θ.

1. (d) Since

$$\log f(y|\theta) = \log 3 + 2\log y - 3\log\theta - \frac{y^3}{\theta^3},$$

we find

$$\frac{\partial}{\partial\theta}\log f(y|\theta) = -\frac{3}{\theta} + \frac{3y^3}{\theta^4} \quad \text{and} \quad \frac{\partial^2}{\partial\theta^2}\log f(y|\theta) = \frac{3}{\theta^2} - \frac{12y^3}{\theta^5}$$

implying

$$I(\theta) = -\mathbb{E}\left[\frac{\partial}{\partial\theta}\log f(Y|\theta)\right] = -\frac{3}{\theta^2} + \frac{12\mathbb{E}(Y^3)}{\theta^5} = \frac{12}{\theta^3}\theta^5 - \frac{3}{\theta^2} = \frac{9}{\theta^2}.$$

1. (e) An approximate 95% confidence interval for θ based on the MLE and Fisher Information is

$$\left[\hat{\theta}_{\text{MLE}} - z_{0.025}\frac{1}{\sqrt{nI(\hat{\theta}_{\text{MLE}})}}\,,\, \hat{\theta}_{\text{MLE}} + z_{0.025}\frac{1}{\sqrt{nI(\hat{\theta}_{\text{MLE}})}}\right]$$

which in this case equals

$$\left[\left(\frac{1}{n}\sum_{i=1}^n Y_i^3\right)^{1/3} - \frac{1.96}{3\sqrt{n}}\left(\frac{1}{n}\sum_{i=1}^n Y_i^3\right)^{1/3}, \left(\frac{1}{n}\sum_{i=1}^n Y_i^3\right)^{1/3} + \frac{1.96}{3\sqrt{n}}\left(\frac{1}{n}\sum_{i=1}^n Y_i^3\right)^{1/3}\right].$$

2. The likelihood ratio for testing $H_0: \theta = \theta_0$ against $H_A: \theta \neq \theta_0$ is

$$\begin{aligned}\Lambda(Y_1,\ldots,Y_n) = \frac{L(\theta_0)}{L(\hat{\theta}_{\text{MLE}})} &= \frac{3^n\theta_0^{-3n}\left(\prod_{i=1}^n Y_i\right)^2 \exp\left\{-\frac{1}{\theta_0^3}\sum_{i=1}^n Y_i^3\right\}}{3^n\hat{\theta}_{\text{MLE}}^{-3n}\left(\prod_{i=1}^n Y_i\right)^2 \exp\left\{-\frac{1}{\hat{\theta}_{\text{MLE}}^3}\sum_{i=1}^n Y_i^3\right\}} \\ &= n^{-n}\theta_0^{-3n}\left(\sum_{i=1}^n Y_i^3\right)^n \exp\left\{n - \frac{1}{\theta_0^3}\sum_{i=1}^n Y_i^3\right\}.\end{aligned}$$

3. Suppose that $X = \theta^{-3}Y^3$. The density function of X is

$$\begin{aligned}f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{P}\left(X \le x\right) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{P}\left(Y \le \theta x^{1/3}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\theta x^{1/3}} \frac{3y^2}{\theta^3}\exp\left\{-\frac{y^3}{\theta^3}\right\}\mathrm{d}y \\ &= \frac{3\theta^2 x^{2/3}}{\theta^3}\exp\left\{-\frac{\theta^3 x}{\theta^3}\right\}\cdot\frac{\theta x^{-2/3}}{3} \\ &= e^{-x}\end{aligned}$$

provided $x > 0$. That is, $X \sim \text{Exp}(1)$. If $a > 0$ and $b > 0$ are chosen so that $\mathbf{P}\left(X < a\right) = \alpha$ and $\mathbf{P}\left(X > b\right) = \alpha$, then a and b satisfy

$$\alpha = \int_0^a e^{-x}\,\mathrm{d}x = 1 - e^{-a} \quad \text{and} \quad \alpha = \int_b^\infty e^{-x}\,\mathrm{d}x = e^{-b}$$

implying that $a = -\log(1-\alpha)$ and $b = -\log\alpha$. Therefore,

$$1 - 2\alpha = \mathbf{P}\left(-\log(1-\alpha) \le X \le -\log\alpha\right) = \mathbf{P}\left(-\log(1-\alpha) \le \frac{Y^3}{\theta^3} \le -\log\alpha\right)$$
$$= \mathbf{P}\left(\frac{Y}{(-\log\alpha)^{1/3}} \le \theta \le \frac{Y}{(-\log(1-\alpha)^{1/3}}\right)$$

so that

$$\left[\frac{Y}{(-\log\alpha)^{1/3}}, \frac{Y}{(-\log(1-\alpha))^{1/3}}\right]$$

is a confidence interval for θ with coverage probability $1 - 2\alpha$.

4. Let $\hat{\theta} = \max Y_1, \ldots, Y_n$. Note that if $0 \le x \le \theta$, then

$$\mathbf{P}\left(\hat{\theta} \le x\right) = \left[\mathbf{P}\left(Y_1 \le \theta\right)\right]^n = \left[\int_0^x \frac{2y}{\theta^2}\,\mathrm{d}y\right]^n = \frac{x^{2n}}{\theta^{2n}}.$$

Therefore, the density function of $\hat{\theta}$ is

$$f_{\hat{\theta}}(x|\theta) = 2n\theta^{-2n}x^{2n-1}, \quad 0 \le x \le \theta.$$

4. (a) Since

$$\mathbb{E}(\hat{\theta}) = \int_0^\theta x \cdot 2n\theta^{-2n}x^{2n-1}\,\mathrm{d}x = \frac{2n}{2n+1}\theta^{-2n}\theta^{2n+1} = \frac{2n}{2n+1}\theta,$$

we conclude

$$\mathrm{bias}(\hat{\theta}) = \mathbb{E}(\hat{\theta}) - \theta = \frac{2n}{2n+1}\theta - \theta = -\frac{1}{2n+1}\theta.$$

4. (b) By definition,

$$\begin{aligned}
\mathrm{MSE}(\hat{\theta}) = \mathbb{E}(\hat{\theta} - \theta)^2 &= \int_0^\theta (x-\theta)^2 \cdot 2n\theta^{-2n}x^{2n-1}\,\mathrm{d}x \\
&= 2n\theta^{-2n}\int_0^\theta (x^2 - 2\theta x + \theta^2)x^{2n-1}\,\mathrm{d}x \\
&= 2n\theta^{-2n}\left[\int_0^\theta x^{2n+1}\,\mathrm{d}x - 2\theta\int_0^\theta x^{2n}\,\mathrm{d}x + \theta^2\int_0^\theta x^{2n-1}\,\mathrm{d}x\right] \\
&= 2n\theta^{-2n}\left[\frac{\theta^{2n+2}}{2n+2} - 2\theta \cdot \frac{\theta^{2n+1}}{2n+1} + \theta^2 \cdot \frac{\theta^{2n}}{2n}\right] \\
&= \frac{\theta^2}{(n+1)(2n+1)}
\end{aligned}$$

5. (a) The likelihood ratio test statistic is

$$\Lambda(y) = \frac{f(y|\theta = 0)}{f(y|\theta = 1/2)} = \frac{1/2}{1/2 - \frac{1/2}{2}(y-1)} = \frac{1}{1 - \frac{1}{2}(y-1)}.$$

The rejection region is

$$\{\Lambda(Y) \le c\} = \left\{\frac{1}{1-\frac{1}{2}(Y-1)} \le c\right\} = \left\{Y \le 3 - \frac{2}{c}\right\} = \{Y \le c'\}$$

where the constant $c' = 3 - 2/c$ is chosen so that the test has significance level α. That is,

$$\alpha = \mathbf{P}\left(Y \le c' \,|\, \theta = 0\right) = \int_0^{c'} \frac{1}{2}\,\mathrm{d}y = \frac{c'}{2}$$

implying $c' = 2\alpha$. Therefore, the rejection region of the significance level α test of $H_0 : \theta = 0$ against $H_A : \theta = 1/2$ is $\{Y \le 2\alpha\}$.

5. (b) The power of this test is

$$\begin{aligned}
\text{power} = \mathbf{P}\left(Y \le 2\alpha \,|\, \theta = 1/2\right) = \int_0^{2\alpha} \left(\frac{1}{2} - \frac{1}{4}(y-1)\right)\mathrm{d}y &= \alpha - \frac{1}{8}\left[(2\alpha-1)^2 - 1\right] \\
&= \alpha - \frac{4\alpha^2 - 4\alpha}{8} \\
&= \frac{\alpha(3-\alpha)}{2}.
\end{aligned}$$

6. (a) Let $U = 1/X$ so that the density function of U is

$$\begin{aligned}
f_U(u) = \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{P}\left(U \le u\right) = \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{P}\left(X \ge 1/u\right) &= \frac{\mathrm{d}}{\mathrm{d}u}(1 - \mathbf{P}\left(X \le 1/u\right)) \\
&= -\frac{\mathrm{d}}{\mathrm{d}u}\int_0^{1/u} \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}\,\mathrm{d}x \\
&= -\frac{b^a}{\Gamma(a)} u^{-(a-1)} e^{-b/u} \cdot -\frac{1}{u^2} \\
&= \frac{b^a}{\Gamma(a)} u^{-a-1} e^{-b/u}
\end{aligned}$$

provided that $u > 0$.

6. (b) The posterior density satisfies

$$f(\theta|y) \propto f(y|\theta) \cdot g(\theta) \propto \frac{1}{\sqrt{2\pi\theta}} \exp\left\{-\frac{y^2}{2\theta}\right\} \cdot \frac{b^a}{\Gamma(a)} \theta^{-a-1} e^{-b/\theta} \propto \theta^{-(a+1/2)-1} e^{-(b+y^2/2)/\theta}$$

which we recognize as the kernel of the density function of an inverse Gamma distribution with parameters $a + 1/2$ and $b + y^2/2$. Thus,

$$f(\theta|y) = \frac{(b+y^2/2)^{a+1/2}}{\Gamma(a+1/2)} \theta^{-(a+1/2)-1} e^{-(b+y^2/2)/\theta}, \quad \theta > 0.$$

7. (a) Since $\log f(y|\theta) = y\log\theta - \log y! - \theta$, we find

$$\frac{\partial}{\partial\theta}\log f(y|\theta) = \frac{y}{\theta} - 1 \quad \text{and} \quad \frac{\partial^2}{\partial\theta^2}\log f(y|\theta) = -\frac{y}{\theta^2}$$

implying

$$I(\theta) = -\mathbb{E}\left[\frac{\partial}{\partial\theta}\log f(Y|\theta)\right] = \frac{\mathbb{E}(Y)}{\theta^2} = \frac{1}{\theta}.$$

7. (b) The Jeffreys prior is

$$J(\theta) = \frac{1}{\sqrt{\theta}}, \quad \theta > 0.$$

Note that $J(\theta)$ is not a legitimate density function. The posterior density function satisfies

$$f(\theta|y) \propto \theta^y e^{-\theta} \cdot \theta^{-1/2} = \theta^{y-1/2}e^{-\theta}$$

for $\theta > 0$ which we recognize as the kernel of a Gamma$(y+1/2, 1)$ random variable. That is,

$$f(\theta|y) = \frac{1}{\Gamma(y+1/2)}\theta^{(y+1/2)-1}e^{-\theta}, \quad \theta > 0.$$

7. (c) The Bayes estimator of θ is simply the posterior mean. That is,

$$\begin{aligned}\hat{\theta}_{\text{BAYES}} = \mathbb{E}[\theta\,|\,y] = \int_0^\infty \theta \cdot \frac{1}{\Gamma(y+1/2)}\theta^{(y+1/2)-1}e^{-\theta}\,\mathrm{d}\theta &= \frac{1}{\Gamma(y+1/2)}\int_0^\infty \theta^{(y+3/2)-1}e^{-\theta}\,\mathrm{d}\theta \\ &= \frac{\Gamma(y+3/2)}{\Gamma(y+1/2)} \\ &= y + 1/2\end{aligned}$$

where the last equality followed from the identity $\Gamma(z+1) = z\Gamma(z)$.

8. The correct answers, in order, are: False, True, False, False, False, False, False, True, True, False, False.