Statistics 252 Winter 2016 Midterm #2 – Solutions

1. (a) The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \frac{2^{n/2}}{\pi^{n/2}}\theta^{n/2}\left(\prod_{i=1}^{n}\frac{1}{y_i}\right)\exp\left\{-\frac{\theta}{2}\sum_{i=1}^{n}\frac{1}{y_i^2}\right\}\mathbf{1}\{y_1 > 0, \dots, y_n > 0\}$$

for $\theta > 0$.

1. (b) If we let $u = \sum_{i=1}^{n}\frac{1}{y_i^2}$, then we can write $L(\theta) = g(u,\theta)\cdot h(y_1,\dots,y_n)$ where $h(y_1,\dots,y_n) = \frac{2^{n/2}}{\pi^{n/2}}\prod_{i=1}^{n}\frac{1}{y_i}$ and $g(u,\theta) = \theta^{n/2}e^{-\theta u/2}$ so by the factorization theorem we conclude that $\sum_{i=1}^{n}\frac{1}{Y_i^2}$ is a sufficient statistic for the estimation of θ.

1. (c) The log-likelihood function is $\ell(\theta) = \frac{n}{2}\log 2 - \frac{n}{2}\log\pi + \frac{n}{2}\log\theta - \sum_{i=1}^{n}\log y_i - \frac{\theta}{2}\sum_{i=1}^{n}\frac{1}{y_i^2}$ implying $\ell'(\theta) = \frac{n}{2\theta} - \frac{1}{2}\sum_{i=1}^{n}\frac{1}{y_i^2}$. Setting $\ell'(\theta) = 0$ and solving for θ implies $\theta = \frac{n}{\sum_{i=1}^{n}\frac{1}{y_i^2}}$. Since $\ell''(\theta) = -\frac{n}{\theta^2} < 0$ for all θ, the second derivative test implies that $\hat{\theta}_{\text{MLE}} = \frac{n}{\sum_{i=1}^{n}\frac{1}{Y_i^2}}$.

1. (d) Since $\log f(y|\theta) = \frac{1}{2}\log 2 - \frac{1}{2}\log\pi + \frac{1}{2}\log\theta - \log y - \frac{\theta}{2y^2}$ we find $\frac{\partial}{\partial\theta}\log f(y|\theta) = \frac{1}{2\theta} - \frac{1}{2y^2}$ and $\frac{\partial^2}{\partial\theta^2}\log f(y|\theta) = -\frac{1}{2\theta^2}$ implying that $I(\theta) = -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f(Y|\theta)\right] = \frac{1}{2\theta^2}$.

1. (e) An approximate 90% confidence interval for θ based on the MLE and Fisher Information is

$$\left[\hat{\theta}_{\text{MLE}} - z_{0.05}\frac{1}{\sqrt{nI(\hat{\theta}_{\text{MLE}})}}\,,\,\hat{\theta}_{\text{MLE}} + z_{0.05}\frac{1}{\sqrt{nI(\hat{\theta}_{\text{MLE}})}}\right]$$

which in this case equals

$$\left[\frac{n}{\sum_{i=1}^{n}\frac{1}{Y_i^2}} - 1.645\cdot\frac{\sqrt{2n}}{\sum_{i=1}^{n}\frac{1}{Y_i^2}}\,,\,\frac{n}{\sum_{i=1}^{n}\frac{1}{Y_i^2}} + 1.645\cdot\frac{\sqrt{2n}}{\sum_{i=1}^{n}\frac{1}{Y_i^2}}\right].$$

2. (a) If Y has density $f(y|\theta)$, then the population mean is $\mathbb{E}(Y) = \int_0^\theta 2\theta^{-2}y^2\,\mathrm{d}y = \frac{2}{3}\theta$. Equating the population mean with the sample mean $\overline{Y}$ implies that $\hat{\theta}_{\mathrm{MOM}} = \frac{3}{2}\overline{Y}$.

2. (b) The likelihood function is

$$L(\theta) = \prod_{i=1}^{3} f(y_i|\theta) = 8\theta^{-6}y_1y_2y_3\,\mathbf{1}\{0 \le \min\{y_1,y_2,y_3\}\}\,\mathbf{1}\{\max\{y_1,y_2,y_3\} \le \theta\}$$

for $\theta > 0$. Since $L(\theta)$ is a strictly decreasing function of θ for $\theta > 0$, and since the support of $L(\theta)$ is $[\max\{y_1,y_2,y_3\},\infty)$, we conclude that the maximum value of θ occurs at the minimum of its support, namely at $\theta = \max\{y_1,y_2,y_3\}$. Thus, $\hat{\theta}_{\mathrm{MLE}} = \max\{Y_1,Y_2,Y_3\}$.

2. (c) Observe that if $0 \le y \le \theta$, then $\mathbf{P}(Y_1 \le y) = \int_0^y 2\theta^{-2}t\,\mathrm{d}t = \theta^{-2}y^2$. If $0 \le x \le \theta$, then $\mathbf{P}\left(\hat{\theta}_{\mathrm{MLE}} \le x\right) = [\mathbf{P}(Y_1 \le x)]^3 = [\theta^{-2}x^2]^3 = \theta^{-6}x^6$. This implies that the distribution function of $\hat{\theta}_{\mathrm{MLE}}$ is

$$F_{\hat{\theta}_{\mathrm{MLE}}}(x) = \begin{cases} 0, & \text{if } x < 0, \\ \theta^{-6}x^6, & \text{if } 0 \le x \le \theta, \\ 1, & \text{if } x > \theta, \end{cases}$$

so that the density function of $\hat{\theta}_{\mathrm{MLE}}$ is $f_{\hat{\theta}_{\mathrm{MLE}}}(x) = 6\theta^{-6}x^5$ if $0 \le x \le \theta$.

2. (d) By definition, $\alpha = P_{H_0}(\text{reject } H_0) = P(\hat{\theta}_{\mathrm{MLE}} > c \,|\, \theta = 1) = 1 - c^6$ using the distribution function computed in **(c)**. This implies that $c = (1-\alpha)^{1/6}$.

2. (e) By definition,

$$\text{power} = P_{H_A}(\text{reject } H_0) = P(\hat{\theta}_{\mathrm{MLE}} > c \,|\, \theta) = 1 - \theta^{-6}c^6 = 1 - \frac{1-\alpha}{\theta^6} = \frac{\theta^6 - 1 + \alpha}{\theta^6}$$

using the distribution function computed in **(c)** and the fact from **(d)** that $c = (1-\alpha)^{1/6}$.