Statistics 252 Winter 2016 Midterm #1 – Solutions

1. (a) Observe that

$$\mathbb{E}(\hat{\theta}) = \mathbb{E}\left(1 - \frac{3}{n}\sum_{i=1}^{n} Y_i^2\right) = 1 - \frac{3}{n}\sum_{i=1}^{n}\mathbb{E}(Y_i^2) = 1 - 3\mathbb{E}(Y_1^2)$$

since $Y_1, \ldots, Y_n$ are i.i.d. Since

$$\mathbb{E}(Y_1^2) = \int_0^1 y^2 \cdot \left[1 - \theta\left(y - \frac{1}{2}\right)\right] \mathrm{d}y = \left[\left(1 + \frac{\theta}{2}\right)\frac{y^3}{3} - \theta\frac{y^4}{4}\right]_{y=0}^{y=1} = \frac{1}{3} + \frac{\theta}{12},$$

we deduce

$$\mathbb{E}(\hat{\theta}) = 1 - 3\left[\frac{1}{3} + \frac{\theta}{12}\right] = \frac{\theta}{4} \quad \text{implying} \quad B(\hat{\theta}) = \mathbb{E}(\hat{\theta}) = \frac{\theta}{4} - \theta = -\frac{3}{4}\theta.$$

1. (b) If $c = 4$, then $\hat{\theta}_1 := 4\hat{\theta}$ satisfies $\mathbb{E}(\hat{\theta}_1) = \mathbb{E}(4\hat{\theta}) = 4\mathbb{E}(\hat{\theta}) = 4 \cdot \frac{\theta}{4} = \theta$ implying that $\hat{\theta}_1$ is an unbiased estimator of θ.

2. (a) If $\hat{\theta} := \min\{Y_1, \ldots, Y_n\}$ and $x > \theta$, then

$$\mathbf{P}\left(\hat{\theta} > x\right) = [\mathbf{P}(Y_1 > x)]^n = \left[\int_x^\infty 2\theta^2 y^{-3}\,\mathrm{d}y\right]^n = -\theta^{2n}x^{-2n}$$

implying that the density for $\hat{\theta}$ is $f_{\hat{\theta}}(x) = 2n\theta^{2n}x^{-2n-1}$, $x > \theta$. Therefore,

$$\mathbb{E}(\hat{\theta}) = \int_\theta^\infty x \cdot 2n\theta^{2n}x^{-2n-1}\,\mathrm{d}x = 2n\theta^{2n}\int_\theta^\infty x^{-2n}\,\mathrm{d}x = 2n\theta^{2n}\frac{\theta^{1-2n}}{2n-1} = \frac{2n}{2n-1}\theta$$

implying that

$$B(\hat{\theta}) = \mathbb{E}(\hat{\theta}) - \theta = \left[\frac{2n}{2n-1} - 1\right]\theta = \frac{1}{2n-1}\theta.$$

2. (b) By definition,

$$\begin{aligned}
\mathrm{MSE}(\hat{\theta}) = \mathbb{E}(\hat{\theta} - \theta)^2 &= \int_\theta^\infty (x-\theta)^2 \cdot 2n\theta^{2n}x^{-2n-1}\,\mathrm{d}x \\
&= 2n\theta^{2n}\left[\int_\theta^\infty x^{1-2n}\,\mathrm{d}x - 2\theta\int_\theta^\infty x^{-2n}\,\mathrm{d}x + \theta^2\int_\theta^\infty x^{-2n-1}\,\mathrm{d}x\right] \\
&= 2n\theta^{2n}\left[\frac{\theta^{2-2n}}{2n-2} - 2\theta\frac{\theta^{1-2n}}{2n-1} + \theta^2\frac{\theta^{-2n}}{2n}\right] \\
&= \theta^2\left[\frac{2n}{2n-2} - \frac{4n}{2n-1} + 1\right] \\
&= \frac{2\theta^2}{(2n-1)(2n-2)} \\
&= \frac{\theta^2}{(n-1)(2n-1)}.
\end{aligned}$$

2. (c) Since $\text{MSE}(\hat{\theta}) = \text{var}(\hat{\theta}) + [B(\hat{\theta})]^2$, we deduce that

$$\text{var}(\hat{\theta}) = \text{MSE}(\hat{\theta}) - [B(\hat{\theta})]^2 = \frac{1}{(n-1)(2n-1)}\theta^2 - \left[\frac{1}{2n-1}\theta\right]^2 = \frac{n\theta^2}{(n-1)(2n-1)^2}$$

and so

$$\sigma_{\hat{\theta}} = \sqrt{\text{var}(\hat{\theta})} = \frac{\theta}{2n-1}\sqrt{\frac{n}{n-1}}.$$

3. We know that

$$\text{Eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\text{var}(\hat{\theta}_2)}{\text{var}(\hat{\theta}_1)},$$

and since $\hat{\theta}_2$ is unbiased, we know that $\text{var}(\hat{\theta}_2) = \text{MSE}(\hat{\theta}_2) = \theta^2/48$. Moreover, we observe that

$$\text{var}(\hat{\theta}_1) = \text{var}\left(\frac{3}{2}\cdot\overline{Y}\right) = \frac{9}{4}\,\text{var}\left(\frac{Y_1+Y_2+Y_3}{3}\right) = \frac{3}{4}\,\text{var}(Y_1)$$

since Y_1, Y_2, Y_3 are i.i.d. Since $\hat{\theta}_1$ is unbiased, we know that

$$\mathbb{E}(\hat{\theta}_1) = \mathbb{E}\left(\frac{3}{2}\cdot\overline{Y}\right) = \theta \qquad \text{which implies that} \qquad \mathbb{E}(\overline{Y}) = \frac{2}{3}\theta.$$

Since Y_1, Y_2, Y_3 are i.i.d., we conclude that $\mathbb{E}(Y_1) = \mathbb{E}(\overline{Y}) = \dfrac{2}{3}\theta$. We also compute

$$\mathbb{E}(Y_1^2) = \int_0^\theta y^2 \cdot 2\theta^{-2} y \,\mathrm{d}y = 2\theta^{-2}\int_0^\theta y^3\,\mathrm{d}y = \frac{\theta^2}{2}$$

so that

$$\text{var}(Y_1) = \mathbb{E}(Y_1^2) - [\mathbb{E}(Y_1)]^2 = \frac{\theta^2}{2} - \left[\frac{2}{3}\theta\right]^2 = \frac{1}{18}\theta^2.$$

This implies

$$\text{var}(\hat{\theta}_1) = \frac{3}{4}\,\text{var}(Y_1) = \frac{3}{4}\cdot\frac{1}{18}\theta^2 = \frac{1}{24}\theta^2$$

and so

$$\text{Eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\text{var}(\hat{\theta}_2)}{\text{var}(\hat{\theta}_1)} = \frac{\theta^2/48}{\theta^2/24} = \frac{24}{48} = \frac{1}{2} < 1.$$

Hence, we conclude that $\hat{\theta}_2$ is preferred to $\hat{\theta}_1$ for the estimation of θ.