Stat 252 Winter 2016
Solutions to Assignment #7

1. (a) Recall that the generalized likelihood ratio test for the simple null hypothesis $H_0 : \theta = \theta_0$ against the composite alternative hypothesis $H_A : \theta \neq \theta_0$ has rejection region $\{\Lambda < c\}$ where

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta}_{\text{MLE}})}$$

is the generalized likelihood ratio and $L(\theta)$ is the likelihood function. In this instance,

$$L(\theta) = \theta^n \exp\left\{-\theta \sum_{i=1}^{n} y_i\right\}$$

so that

$$\Lambda = \frac{\theta_0^n \exp\left\{-\theta_0 \sum_{i=1}^{n} y_i\right\}}{\hat{\theta}_{\text{MLE}}^n \exp\left\{-\hat{\theta}_{\text{MLE}} \sum_{i=1}^{n} y_i\right\}} = \left(\frac{\theta_0}{1/\overline{Y}}\right)^n \exp\left\{-\theta_0 \sum y_i + 1/\overline{Y} \cdot \sum y_i\right\}$$
$$= \left(\theta_0 \overline{Y}\right)^n \exp\left\{n - n\theta_0\overline{Y}\right\} = e^n\, \theta_0^n\, \overline{Y}^{\,n} \exp\left\{-n\theta_0\overline{Y}\right\} = e^n\, \theta_0^n \left[\overline{Y} \exp\left\{-\theta_0\overline{Y}\right\}\right]^n.$$

Hence, we see that the rejection region $\{\Lambda < c\}$ can be expressed as

$$\left\{e^n\, \theta_0^n \left[\overline{Y} \exp\left\{-\theta_0\overline{Y}\right\}\right]^n < c\right\} = \left\{\overline{Y} \exp\left\{-\theta_0\overline{Y}\right\} < c^{1/n} e^{-1}\, \theta_0^{-1}\right\}$$
$$= \left\{\overline{Y} \exp\left\{-\theta_0\overline{Y}\right\} < C\right\}.$$

(To be explicit, the *suitable constant* is $C = c^{1/n} e^{-1}\, \theta_0^{-1}$.)

1. (b) We saw in class that $-2\log\Lambda \sim \chi^2(1)$ (approximately). This means that the generalized likelihood ratio test rejection region is $\{\Lambda < c\} = \{-2\log\Lambda > K\}$ where K is (yet another) constant. As we found above,

$$\Lambda = e^n\, \theta_0^n \left[\overline{Y} \exp\left\{-\theta_0\overline{Y}\right\}\right]^n$$

so that

$$-2\log\Lambda = -2n - 2n\log\theta_0 - 2n\log\overline{Y} + 2n\theta_0\overline{Y}.$$

Hence, to conduct the GLRT, we need to compare the observed value of $-2\log\Lambda$ with the appropriate chi-squared critical value which is $\chi^2_{0.10,1} = 2.70554$. Since

$$-2 \cdot 10 - 2 \cdot 10 \log 1 - 2 \cdot 10 \cdot \log 1.25 + 2 \cdot 10 \cdot 1 \cdot 1.25 \approx 2.76856$$

is the observed value of $-2\log\Lambda$, we reject H_0 at significance level 0.10. (Note, however, that since $\chi^2_{0.05,1} = 3.84146$, we fail to reject H_0 at significance level 0.05.)

2. (a) Recall that the generalized likelihood ratio test for the simple null hypothesis $H_0 : \theta = \theta_0$ against the composite alternative hypothesis $H_A : \theta \neq \theta_0$ has rejection region $\{\Lambda < c\}$ where

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta}_{\text{MLE}})}$$

is the generalized likelihood ratio and $L(\theta)$ is the likelihood function. In this instance,

$$L(\theta) = \theta^{2n} \left(\prod_{i=1}^{n} y_i \right) \exp\left\{ -\theta \sum_{i=1}^{n} y_i \right\}$$

so that

$$\Lambda = \frac{\theta_0^{2n} \left(\prod_{i=1}^{n} y_i \right) \exp\left\{ -\theta_0 \sum_{i=1}^{n} y_i \right\}}{\hat{\theta}_{\text{MLE}}^{2n} \left(\prod_{i=1}^{n} y_i \right) \exp\left\{ -\hat{\theta}_{\text{MLE}} \sum_{i=1}^{n} y_i \right\}} = \left(\frac{1}{2/\overline{Y}} \right)^{2n} \exp\left\{ -\sum y_i + 2/\overline{Y} \cdot \sum y_i \right\}$$

$$= \left(\frac{\overline{Y}}{2} \right)^{2n} \exp\left\{ 2n - n\overline{Y} \right\}.$$

2. (b) We saw in class that $-2\log\Lambda \sim \chi^2(1)$ (approximately). This means that the generalized likelihood ratio test rejection region is $\{\Lambda < c\} = \{-2\log\Lambda > K\}$ where $K = -2\log c$ is (yet another) constant. (In fact, $K = \chi^2_{\alpha,1}$.) As we found above,

$$\Lambda = \left(\frac{\overline{Y}}{2} \right)^{2n} \exp\left\{ 2n - n\overline{Y} \right\}$$

so that

$$-2\log\Lambda = -4n\log\overline{Y} + 4n\log 2 - 4n + 2n\overline{Y}.$$

Hence, to conduct the GLRT, we need to compare the observed value of $-2\log\Lambda$ with the appropriate chi-squared critical value which is $\chi^2_{0.05,1} = 3.84146$. Since

$$-4 \cdot 5 \cdot \log 1 + 4 \cdot 5 \cdot \log 2 - 4 \cdot 5 + 2 \cdot 5 \cdot 1 \approx 3.8629$$

is the observed value of $-2\log\Lambda$, we reject H_0 at significance level 0.05 (but just barely).

3. Suppose that $Y \sim \text{Bin}(n, \theta)$ with $\theta \sim \beta(a, b)$ so that

$$g(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}(1-\theta)^{b-1}, \quad 0 \leq \theta \leq 1.$$

The posterior density satisfies

$$f(\theta|y) \propto f(y|\theta)g(\theta) \propto \theta^y(1-\theta)^{n-y}\theta^{a-1}(1-\theta)^{b-1} = \theta^{y+a-1}(1-\theta)^{n-y+b-1}$$

from which we conclude that the posterior distribution of θ given y is $\beta(y+a, n-y+b)$.

4. If $Y_1, \dots Y_n$ are i.i.d. Poisson(θ) random variables, then

$$L(\theta) = f(y_1, \dots, y_n|\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \frac{1}{y_1! \cdots y_n!} e^{-n\theta} \theta^{y_1 + \cdots + y_n}.$$

If the prior distribution of θ is $\Gamma(\alpha, \beta)$, then

$$f(\theta|y_1, \dots, y_n) \propto f(y_1, \dots, y_n \theta) g(\theta) \propto e^{-n\theta} \theta^{y_1 + \cdots + y_n} \cdot \theta^{\alpha - 1} e^{-\theta/\beta} = e^{-\theta(n+1/\beta)} \theta^{\alpha - 1 + y_1 + \cdots + y_n}$$

which implies that the posterior distribution of θ given $(y_1, \dots, y_n)$ is

$$\Gamma\left(\alpha + \sum_{i=1}^{n} y_i \ , \ \frac{1}{n + 1/\beta}\right).$$

5. (a) An expression for the posterior density is

$$f(\theta|y) = \frac{f(y|\theta)g(\theta)}{\displaystyle\int_{-\infty}^{\infty} f(y|\theta)g(\theta)\,\mathrm{d}\theta} = \frac{\displaystyle\frac{1}{\theta}\exp\{-\theta^2 - y^2/\theta\}}{\displaystyle\int_{0}^{\infty} \frac{1}{\theta}\exp\{-\theta^2 - y^2/\theta\}\,\mathrm{d}\theta}.$$

5. (b) If $y = 1$, then

$$f(\theta|y=1) = \frac{\displaystyle\frac{1}{\theta}\exp\{-\theta^2 - 1/\theta\}}{\displaystyle\int_{0}^{\infty} \frac{1}{\theta}\exp\{-\theta^2 - 1/\theta\}\,\mathrm{d}\theta}.$$

Using MAPLE

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> evalf(Int(exp(-x^2-1/x)/x, x=0..infinity));
>                0.1869287323
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we find

$$\int_{0}^{\infty} \frac{1}{\theta}\exp\{-\theta^2 - 1/\theta\}\,\mathrm{d}\theta = 0.1869287323$$

and so

$$f(\theta|y=1) = \frac{1}{0.1869287323\theta} \cdot \exp\{-\theta^2 - 1/\theta\} = \frac{5.349632385}{\theta} \cdot \exp\{-\theta^2 - 1/\theta\}, \quad \theta > 0.$$