Stat 252 Winter 2016
Solutions to Assignment #6

1. (a) Since $\log f(y|\theta) = y\log(\theta) - y - \log(y!)$ we find

$$\frac{\partial}{\partial\theta}\log f(y|\theta) = \frac{y}{\theta} \quad \text{and} \quad \frac{\partial^2}{\partial\theta^2}\log f(y|\theta) = -\frac{y}{\theta^2}.$$

Thus,

$$I(\theta) = -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\log f(Y|\theta)\right) = \frac{\mathbb{E}(Y)}{\theta^2} = \frac{\theta}{\theta^2} = \frac{1}{\theta}.$$

1. (b) If $Y \sim \text{Poisson}(\theta)$, then since $\mathbb{E}(Y) = \theta$, setting $\mathbb{E}(Y) = \overline{Y}$ gives $\hat{\theta}_{\text{MOM}} = \overline{Y}$.

1. (c) Since $\mathbb{E}(Y_1) = \theta$, we conclude that

$$\mathbb{E}(\hat{\theta}_{\text{MOM}}) = \mathbb{E}(\overline{Y}) = \mathbb{E}\left(\frac{Y_1 + \cdots + Y_n}{n}\right) = \mathbb{E}(Y_1) = \theta$$

so that $\hat{\theta}_{\text{MOM}}$ is an unbiased estimator of θ.

1. (d) Since $\text{Var}(Y_1) = \theta$, and since the Y_i are i.i.d., we conclude

$$\text{Var}(\hat{\theta}_{\text{MOM}}) = \text{Var}(\overline{Y}) = \text{Var}\left(\frac{Y_1 + \cdots + Y_n}{n}\right) = \frac{\text{Var}(Y_1)}{n} = \frac{\theta}{n}.$$

1. (e) The Cramer-Rao inequality tells us that an unbiased estimator $\hat{\theta}$ of θ must satisfy

$$\text{Var}(\hat{\theta}) \geq \frac{1}{nI(\theta)} = \frac{\theta}{n}$$

since we found in **(a)** that $I(\theta) = 1/\theta$. From **(c)** we know that $\hat{\theta}_{\text{MOM}}$ is unbiased, and from **(d)** we know that $\text{Var}(\hat{\theta}_{\text{MOM}}) = \theta/n$. Since we have found an unbiased estimator, namely $\hat{\theta}_{\text{MOM}}$, whose variance attains the lower bound of the Cramer-Rao inequality, we conclude that $\hat{\theta}_{\text{MOM}}$ must be the MVUE of θ.

2. (a) By definition, the significance level α is the probability of a Type I error; that is, the probability under H_0 that H_0 is rejected. Hence, since $\frac{4S^2}{\sigma^2} \sim \chi^2(4)$, we conclude

$$\begin{aligned}\alpha = P_{H_0}(\text{reject } H_0) = P(S^2 > 1.945|\sigma^2 = 1) &= P\left(\frac{4S^2}{1} > \frac{4 \cdot 1.945}{1}\right) \\ &= P(X > 7.78) \doteq 0.10,\end{aligned}$$

where $X \sim \chi^2(4)$. (The last step follows from a table of chi-squared values.) Hence, we see that the hypothesis test does, in fact, have significance level $\alpha = 0.10$.

2. (b) By definition, the power of a test is the probability under H_A that H_0 is rejected. Hence, when $\sigma = 3.3$, we find

$$\begin{aligned}\text{power} = P_{H_A}(\text{reject } H_0) = P(S^2 > 1.945|\sigma^2 = 3.3^2) &= P\left(\frac{4S^2}{3.3^2} > \frac{4 \cdot 1.945}{3.3^2}\right) \\ &\doteq P(X > 0.71) \doteq 0.95,\end{aligned}$$

where $X \sim \chi^2(4)$. (The last step follows from a table of chi-squared values.) Hence, the power of this test when $\sigma = 3.3$ is 0.95.

3. (a) Since the likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \theta^{2n} \left(\prod_{i=1}^{n} y_i \right)^{-3n} \exp\left\{ -\theta \sum_{i=1}^{n} \frac{1}{y_i} \right\},$$

if we let $u = \sum_{i=1}^{n} \frac{1}{y_i}$, then we can write $L(\theta) = g(u,\theta) \cdot h(y_1, \ldots, y_n)$ where

$$h(y_1, \ldots, y_n) = \left(\prod_{i=1}^{n} y_i \right)^{-3n} \quad \text{and} \quad g(u,\theta) = \theta^{2n} \exp\{-\theta u\}$$

so by the Factorization Theorem we conclude that $\sum_{i=1}^{n} \frac{1}{Y_i}$ is a sufficient statistic for the estimation of θ.

3. (b) Recall from class that any one-to-one function of a sufficient statistic is also sufficient. Therefore, if we let

$$T(U) = \frac{2n}{U},$$

then since T is one-to-one, we find that

$$T\left(\sum_{i=1}^{n} \frac{1}{Y_i} \right) = \frac{2n}{\sum_{i=1}^{n} \frac{1}{Y_i}} = \hat{\theta}_{\text{MLE}}$$

is also a sufficient statistic for the estimation of θ.

3. (c) Since $\log f(y|\theta) = 2\log(\theta) - 3\log(y) - \frac{\theta}{y}$, we find

$$\frac{\partial}{\partial\theta} \log f(y|\theta) = \frac{2}{\theta} - \frac{1}{y} \quad \text{and} \quad \frac{\partial^2}{\partial\theta^2} \log f(y|\theta) = -\frac{2}{\theta^2}.$$

Thus,

$$I(\theta) = -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2} \log f(Y|\theta) \right) = \frac{2}{\theta^2}.$$

3. (d) An approximate 90% confidence interval for θ based on the MLE and Fisher Information is

$$\left[\hat{\theta}_{\text{MLE}} - z_{0.05} \frac{1}{\sqrt{nI(\hat{\theta}_{\text{MLE}})}} , \hat{\theta}_{\text{MLE}} + z_{0.05} \frac{1}{\sqrt{nI(\hat{\theta}_{\text{MLE}})}} \right]$$

which in this case equals

$$\left[\frac{2n}{\sum_{i=1}^{n} \frac{1}{Y_i}} - 1.645 \cdot \frac{\sqrt{2n}}{\sum_{i=1}^{n} \frac{1}{Y_i}} , \frac{2n}{\sum_{i=1}^{n} \frac{1}{Y_i}} + 1.645 \cdot \frac{\sqrt{2n}}{\sum_{i=1}^{n} \frac{1}{Y_i}} \right].$$

3. (e) The rejection region of a significance level 0.10 test of $H_0 : \theta = \theta_0$ vs. $H_A : \theta \neq \theta_0$ based on the Fisher information and the MLE is

$$\left\{ \sqrt{nI(\hat{\theta}_{\text{MLE}})} \left| \hat{\theta}_{\text{MLE}} - \theta_0 \right| > z_{0.05} \right\} \quad \text{which in this case equals} \quad \left\{ \left| \sqrt{2n} - \frac{\theta_0}{\sqrt{2n}} \sum_{i=1}^{n} \frac{1}{Y_i} \right| > 1.645 \right\}.$$

4. As a result of the confidence interval–hypothesis test duality, we know that the rejection region for the level 0.10 test of $H_0 : \theta = 4$ vs. $H_A : \theta \neq 4$ is $RR = \{4 \notin (Y - 2, Y + 3)\}$. That is, we reject H_0 in favour of H_A if $4 < Y - 2$ or $Y + 3 < 4$. In other words, $RR = \{Y < 1 \text{ or } Y > 6\}$.

5. (a) By definition, the significance level α is the probability of a Type I error; that is, the probability under H_0 that H_0 is rejected, or $\alpha = P_{H_0}(\text{reject } H_0) = P(Y > c \,|\, \theta = 1)$. If we assume that Y is Uniform$(0, 1)$, then $P(Y > c) = 1 - c$ so that in order to have a significance level 0.05 test, we need $c = 0.95$.

5. (b) By definition, the power of a test is the probability under H_A that H_0 is rejected. That is, power $= P_{H_A}(\text{reject } H_0) = P(Y > 0.95 \,|\, \theta)$. If we assume that Y is Uniform$[0, \theta]$, then

$$P(Y > 0.95 \,|\, \theta) = \frac{\theta - 0.95}{\theta} = 1 - \frac{19}{20\,\theta}.$$