Stat 252 Winter 2016
Solutions to Assignment #5

1. Let $U = \theta^2 Y$ so that for $u > 0$,

$$P(U \le u) = P(Y \le \theta^{-2}u) = \int_0^{\theta^{-2}u} \theta^2 e^{-\theta^2 y}\, \mathrm{d}y$$

which implies that the density function of U is therefore

$$f_U(u) = \frac{\mathrm{d}}{\mathrm{d}u} \int_0^{\theta^{-2}u} \theta^2 e^{-\theta^2 y}\, \mathrm{d}y = e^{-u}$$

for $u > 0$. Thus, we must find a and b so that

$$\alpha_1 = P(U < a) = \int_0^a e^{-u}\, \mathrm{d}u \quad \text{and} \quad \alpha_2 = P(U > b) = \int_b^\infty e^{-u}\, \mathrm{d}u.$$

Computing the integrals we find $1 - e^{-a} = \alpha_1$ and $e^{-b} = \alpha_2$. Hence,

$$\begin{aligned} 1 - \alpha = P(a \le U \le b) &= P\big(-\log(1-\alpha_1) \le \theta^2 Y \le -\log(\alpha_2)\big) \\ &= P\left(\sqrt{\frac{-\log(1-\alpha_1)}{Y}} \le \theta \le \sqrt{\frac{-\log(\alpha_2)}{Y}}\right). \end{aligned}$$

In other words,

$$\left[\sqrt{\frac{-\log(1-\alpha_1)}{Y}}, \sqrt{\frac{-\log(\alpha_2)}{Y}}\right]$$

is a confidence interval for θ with coverage probability $1 - (\alpha_1 + \alpha_2)$.

2. Let $U = Y/\theta$ so that for $0 \le u \le 1$,

$$P(U \le u) = P(Y \le \theta u) = \int_0^{\theta u} 2\theta^{-2} y\, dy = \frac{y^2}{\theta^2}\bigg|_0^{\theta u} = u^2.$$

The density function of U is therefore $f_U(u) = 2u$ for $0 \le u \le 1$. Thus, we must find a and b so that

$$\frac{\alpha}{2} = P(U < a) = \int_0^a 2u\, \mathrm{d}u \quad \text{and} \quad \frac{\alpha}{2} = P(U > b) = \int_b^1 2u\, \mathrm{d}u.$$

Computing the integrals we find $a^2 = \alpha/2$ and $1 - b^2 = \alpha/2$. Hence,

$$\begin{aligned} 1 - \alpha = P(a \le U \le b) &= P\left(\sqrt{\alpha/2} \le \frac{Y}{\theta} \le \sqrt{1-\alpha/2}\right) \\ &= P\left(\frac{Y}{\sqrt{1-\alpha/2}} \le \theta \le \frac{Y}{\sqrt{\alpha/2}}\right). \end{aligned}$$

In other words,

$$\left[\frac{Y}{\sqrt{1-\alpha/2}}, \frac{Y}{\sqrt{\alpha/2}}\right]$$

is a confidence interval for θ with coverage probability $1 - \alpha$.

3. Let $U = Y - \theta$ so that for $-\infty < u < \infty$,

$$P(U \le u) = P(Y \le \theta + u) = \int_{-\infty}^{\theta+u} \frac{e^{(y-\theta)}}{\left[1+e^{(y-\theta)}\right]^2}\,\mathrm{d}y = -\frac{1}{1+e^{(y-\theta)}}\bigg|_{-\infty}^{\theta+u} = 1 - \frac{1}{1+e^u}.$$

The density function of U is therefore $f_U(u) = \frac{e^u}{(1+e^u)^2}$ for $-\infty < u < \infty$. Thus, we must find a and b so that

$$\alpha_1 = P(U < a) = \int_{-\infty}^{a} \frac{e^u}{(1+e^u)^2}\,\mathrm{d}u \quad \text{and} \quad \alpha_2 = P(U > b) = \int_{b}^{\infty} \frac{e^u}{(1+e^u)^2}\,\mathrm{d}u.$$

Computing the integrals we find

$$\alpha_1 = 1 - \frac{1}{1+e^a} \quad \text{and} \quad \alpha_2 = \frac{1}{1+e^b}$$

and so solving for a and b we find

$$a = \log\left(\frac{\alpha_1}{1-\alpha_1}\right) \quad \text{and} \quad b = \log\left(\frac{1-\alpha_2}{\alpha_2}\right).$$

Hence,

$$\begin{aligned} 1-\alpha = P(a \le U \le b) &= P\left(\log\left(\frac{\alpha_1}{1-\alpha_1}\right) \le Y - \theta \le \log\left(\frac{1-\alpha_2}{\alpha_2}\right)\right) \\ &= P\left(Y - \log\left(\frac{1-\alpha_2}{\alpha_2}\right) \le \theta \le Y - \log\left(\frac{\alpha_1}{1-\alpha_1}\right)\right). \end{aligned}$$

In other words,

$$\left[\ Y - \log\left(\frac{1-\alpha_2}{\alpha_2}\right),\ Y - \log\left(\frac{\alpha_1}{1-\alpha_1}\right)\ \right]$$

is a confidence interval for θ with coverage probability $1 - (\alpha_1 + \alpha_2)$.

4. Since $Y_1, \ldots, Y_n$ are iid Uniform$[0, \theta]$ random variables, we know that their common density is $f(y|\theta) = \theta^{-1}$ for $0 \le y \le \theta$. This implies that if $0 \le x \le \theta$, then $\mathbf{P}\left(\hat{\theta} \le x\right) = \left[\mathbf{P}\left(Y_1 \le x\right)\right]^n = \theta^{-n}x^n$. Suppose that $U = \theta^{-1}\hat{\theta}$ so that if $0 \le u \le 1$, then $\mathbf{P}\left(U \le u\right) = \mathbf{P}\left(\hat{\theta} \le \theta u\right) = \theta^{-n}(\theta u)^n = u^n$. This implies that the density function of U is $f_U(u) = nu^{n-1}$ for $0 \le u \le 1$. Observe that

$$\int_0^a nu^{n-1}\,\mathrm{d}u = a^n \quad \text{and} \quad \int_b^1 nu^{n-1}\,\mathrm{d}u = 1 - b^n.$$

If we choose a and b to satisfy $a^n = 0.05$ and $1 - b^n = 0.05$ so that $a = (0.05)^{1/n}$ and $b = (0.95)^n$, then

$$0.90 = \mathbf{P}\left((0.05)^{1/n} \le U \le (0.95)^{1/n}\right) = \mathbf{P}\left((0.05)^{1/n} \le \frac{\hat{\theta}}{\theta} \le (0.95)^{1/n}\right) = \mathbf{P}\left(\frac{\hat{\theta}}{(0.95)^{1/n}} \le \theta \le \frac{\hat{\theta}}{(0.05)^{1/n}}\right).$$

Thus, a 90% confidence interval for θ based on $\hat{\theta}$ is

$$\left[\frac{\hat{\theta}}{(0.95)^{1/n}}, \frac{\hat{\theta}}{(0.05)^{1/n}}\right].$$