Stat 252 Winter 2016
Solutions to Assignment #4

1. The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \theta^{2n} \left(\prod_{i=1}^{n} y_i\right)^{-3n} \exp\left\{-\theta \sum_{i=1}^{n} \frac{1}{y_i}\right\}$$

provided that $\theta > 0$ and $\min\{y_1, \ldots, y_n\} > 0$, and so the log-likelihood function is

$$\ell(\theta) = 2n\log(\theta) - 3n\sum_{i=1}^{n} \log(y_i) - \theta \sum_{i=1}^{n} \frac{1}{y_i}, \quad \theta > 0,\ \min\{y_1, \ldots, y_n\} > 0.$$

Note that

$$\ell'(\theta) = \frac{2n}{\theta} - \sum_{i=1}^{n} \frac{1}{y_i} \quad \text{and} \quad \ell''(\theta) = -\frac{2n}{\theta^2}.$$

Hence $\ell'(\theta) = 0$ implies

$$0 = \frac{2n}{\theta} - \sum_{i=1}^{n} \frac{1}{y_i} \quad \text{and so} \quad \theta = \frac{2n}{\displaystyle\sum_{i=1}^{n} \frac{1}{y_i}}.$$

Since $\ell''(\theta) < 0$ for all θ, we conclude from the second derivative test that

$$\hat{\theta}_{\text{MLE}} = \frac{2n}{\displaystyle\sum_{i=1}^{n} \frac{1}{Y_i}}.$$

2. (a) To find $\hat{\theta}_{\text{MOM}}$ we solve the equation $\mathbb{E}(Y) = \overline{Y}$ for θ. Since

$$\mathbb{E}(Y) = \int_0^1 y f(y|\theta)\,\mathrm{d}y = (\theta+1)\int_0^1 y^{\theta+1}\,\mathrm{d}y = \left(\frac{\theta+1}{\theta+2}\right) y^{\theta+2}\Big|_0^1 = \frac{\theta+1}{\theta+2}$$

we conclude that

$$\frac{\theta+1}{\theta+2} = \overline{Y}.$$

Solving for θ yields

$$\hat{\theta}_{\text{MOM}} = \frac{2\overline{Y} - 1}{1 - \overline{Y}}.$$

2. (b) The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(y_i|\theta) = (\theta+1)^n \left(\prod_{i=1}^{n} y_i\right)^{\theta}$$

provided that $\theta > 0$ and $0 \le \min\{y_1, \ldots, y_n\} \le \max\{y_1, \ldots, y_n\} \le 1$, and so the log-likelihood function is

$$\ell(\theta) = n\log(\theta+1) + \theta\sum_{i=1}^{n} \log(y_i), \quad \theta > 0,\ 0 \le \min\{y_1, \ldots, y_n\} \le \max\{y_1, \ldots, y_n\} \le 1.$$

Note that

$$\ell'(\theta) = \frac{n}{\theta+1} + \sum_{i=1}^{n} \log(y_i) \quad \text{and} \quad \ell''(\theta) = -\frac{n}{(\theta+1)^2}.$$

Hence $\ell'(\theta) = 0$ implies

$$0 = \frac{n}{\theta+1} + \sum_{i=1}^{n} \log(y_i) \quad \text{and so} \quad \theta = -\frac{n}{\sum_{i=1}^{n} \log(y_i)} - 1.$$

Since $\ell''(\theta) < 0$ for all θ, we conclude from the second derivative test that

$$\hat{\theta}_{\text{MLE}} = -\frac{n}{\sum_{i=1}^{n} \log(Y_i)} - 1.$$

3. (a) Since

$$\log f(y|\theta) = \log y - 2\log\theta - \frac{y^2}{2\theta^2},$$

we find

$$\frac{\partial}{\partial\theta} \log f(y|\theta) = -\frac{2}{\theta} + \frac{y^2}{\theta^3} \quad \text{and} \quad \frac{\partial^2}{\partial\theta^2} \log f(y|\theta) = \frac{2}{\theta^2} - \frac{3y^2}{\theta^4}.$$

Thus, we conclude that

$$I(\theta) = -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2} \log f(Y|\theta)\right) = \frac{3\mathbb{E}(Y^2)}{\theta^4} - \frac{2}{\theta^2} = \frac{4}{\theta^2}$$

since $\mathbb{E}(Y^2) = 2\theta^2$.

3. (b) To find $\hat{\theta}_{\text{MOM}}$ we solve the equation $\mathbb{E}(Y) = \overline{Y}$ for θ. Since $\mathbb{E}(Y) = \sqrt{(\pi/2)}\,\theta$, this implies

$$\hat{\theta}_{\text{MOM}} = \sqrt{\frac{2}{\pi}}\,\overline{Y}.$$

3. (c) We find

$$\begin{aligned}
\text{Var}(\hat{\theta}_{\text{MOM}}) = \text{Var}\left(\sqrt{\frac{2}{\pi}}\,\overline{Y}\right) = \frac{2}{\pi}\ \text{Var}(\overline{Y}) &= \frac{2}{n\pi}\ \text{Var}(Y_1) = \frac{2}{n\pi}\ \left(\mathbb{E}(Y_1^2) - [\mathbb{E}(Y)]^2\right) \\
&= \frac{2}{n\pi}\ \left(2 - \frac{\pi}{2}\right)\theta^2 \\
&= \left(\frac{4-\pi}{n\pi}\right)\ \theta^2.
\end{aligned}$$

3. (d) The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \left(\prod_{i=1}^{n} y_i\right) \theta^{-2n} \exp\left\{-\frac{1}{2\theta^2}\sum_{i=1}^{n} y_i^2\right\}$$

provided that $\theta > 0$ and $\min\{y_1, \ldots, y_n\} > 0$ so that the log-likelihood function is

$$\ell(\theta) = \sum_{i=1}^{n} \log y_i - 2n\log\theta - \frac{1}{2\theta^2}\sum_{i=1}^{n} y_i^2, \quad \theta > 0, \ \min\{y_1, \ldots, y_n\} > 0.$$

Note that

$$\ell'(\theta) = -\frac{2n}{\theta} + \frac{1}{\theta^3}\sum_{i=1}^{n} y_i^2 \quad \text{and} \quad \ell''(\theta) = \frac{2n}{\theta^2} - \frac{3}{\theta^4}\sum_{i=1}^{n} y_i^2.$$

Hence $\ell'(\theta) = 0$ implies

$$0 = -\frac{2n}{\theta} + \frac{1}{\theta^3}\sum_{i=1}^{n} y_i^2 \quad \text{and so} \quad \theta = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} y_i^2}.$$

Since

$$\ell''\left(\sqrt{\frac{1}{2n}\sum_{i=1}^{n} y_i^2}\right) = -\frac{8n^2}{\sum_{i=1}^{n} y_i^2} < 0,$$

we conclude from the second derivative test that

$$\hat{\theta}_{\text{MLE}} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} Y_i^2}.$$

4. The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \frac{1}{(2\theta+1)^n}$$

provided that $\theta > -1/2$ and $0 \le \min\{y_1, \ldots, y_n\} \le \max\{y_1, \ldots, y_n\} \le 2\theta + 1$. Since the support of the density $f(y|\theta)$ depends on θ, we know that we will not be able to use the second derivative test to maximize the likelihood function. However, since $L(\theta)$ is a strictly decreasing function for $\theta > -1/2$, we conclude that the maximum value of $L(\theta)$ occurs when $\max\{y_1, \ldots, y_n\} = 2\theta + 1$. In other words,

$$\hat{\theta}_{\text{MLE}} = \frac{\max\{Y_1, \ldots, Y_n\} - 1}{2}.$$