Stat 252 Winter 2016
Solutions to Assignment #3

1. Since $\log f(y|\theta) = 2\log\theta - \theta^2 y$, we find

$$\frac{\partial}{\partial\theta}\log f(y|\theta) = \frac{2}{\theta} - 2\theta y \quad \text{and} \quad \frac{\partial^2}{\partial\theta^2}\log f(y|\theta) = -\frac{2}{\theta^2} - 2y.$$

Noting that $Y \sim \text{Exp}(\theta^{-2})$ so that $\mathbb{E}(Y) = \theta^{-2}$, we conclude that

$$I(\theta) = -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\log f(Y|\theta)\right) = \frac{2}{\theta^2} + 2\mathbb{E}(Y) = \frac{2}{\theta^2} + \frac{2}{\theta^2} = \frac{4}{\theta^2}.$$

2. Since $\log f(y|\theta) = 2\log\theta - 3\log y - \theta y^{-1}$, we find

$$\frac{\partial}{\partial\theta}\log f(y|\theta) = \frac{2}{\theta} - \frac{1}{y} \quad \text{and} \quad \frac{\partial^2}{\partial\theta^2}\log f(y|\theta) = -\frac{2}{\theta^2}.$$

Thus, we conclude that

$$I(\theta) = -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\log f(Y|\theta)\right) = \frac{2}{\theta^2}.$$

3. (a) If $Y \sim f(y|\theta)$, then

$$\mathbb{E}(Y) = \int_0^\infty \frac{y^3}{2\theta^3}\exp\{-y/\theta\}\,\mathrm{d}y = \frac{\theta}{2}\int_0^\infty u^3 e^{-u}\,\mathrm{d}u = \frac{\theta}{2}\Gamma(4) = \frac{3!\theta}{2} = 3\theta.$$

Therefore, $\mathbb{E}(\overline{Y}) = 3\theta$ since the expected value of the sample mean is always equal to the population mean, and so

$$\mathbb{E}(\hat{\theta}) = \frac{\mathbb{E}(\overline{Y})}{3} = \frac{3\theta}{3} = \theta$$

implying that $\hat{\theta}$ is an unbiased estimator of θ.

3. (b) Since

$$\log f(y|\theta) = -\log 2 + 2\log y - 3\log\theta - \frac{y}{\theta},$$

we find

$$\frac{\partial}{\partial\theta}\log f(y|\theta) = -\frac{3}{\theta} + \frac{y}{\theta^2} \quad \text{and} \quad \frac{\partial^2}{\partial\theta^2}\log f(y|\theta) = \frac{3}{\theta^2} - \frac{2y}{\theta^3},$$

which implies that

$$I(\theta) = -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\log f(Y|\theta)\right) = -\frac{3}{\theta^2} + \frac{2\mathbb{E}(Y)}{\theta^3} = -\frac{3}{\theta^2} + \frac{2\cdot 3\theta}{\theta^3} = \frac{3}{\theta^2}.$$

3. (c) We being by noting that if $Y \sim f(y|\theta)$, then

$$\mathbb{E}(Y^2) = \int_0^\infty \frac{y^4}{2\theta^3} \exp\{-y/\theta\} \, \mathrm{d}y = \frac{\theta^2}{2} \int_0^\infty u^4 e^{-u} \, \mathrm{d}u = \frac{\theta^2}{2}\Gamma(5) = \frac{4!\theta^2}{2} = 12\theta^2$$

implying that $\mathrm{Var}(Y) = 12\theta^2 - (3\theta)^2 = 3\theta^2$. The Cramér-Rao inequality tells us that any unbiased estimator $\hat{\theta}$ of θ must satisfy

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{nI(\theta)} = \frac{\theta^2}{3n}.$$

Since

$$\mathrm{Var}(\hat{\theta}) = \frac{1}{3^2} \mathrm{Var}(\overline{Y}) = \frac{1}{3^2} \cdot \frac{3\theta^2}{n} = \frac{\theta^2}{3n},$$

we have found an unbiased estimator whose variance attains the lower bound of the Cramér-Rao inequality. Hence, $\hat{\theta}$ must be the MVUE of θ.

4. (a) We know that $\mathbb{E}(\overline{Y}) = 252\theta$ since the expected value of the sample mean is always equal to the population mean. Thus, if $\hat{\theta}_1 = \overline{Y}/252$, then $\hat{\theta}_1$ is an unbiased estimator of θ.

4. (b) Since

$$\log f(y|\theta) = -252 \log \theta - \log(251!) + 251 \log y - \frac{y}{\theta},$$

we find

$$\frac{\partial}{\partial \theta} \log f(y|\theta) = -\frac{252}{\theta} + \frac{y}{\theta^2} \quad \text{and} \quad \frac{\partial^2}{\partial \theta^2} \log f(y|\theta) = \frac{252}{\theta^2} - \frac{2y}{\theta^3},$$

which implies that

$$I(\theta) = -\mathbb{E}\left(\frac{\partial^2}{\partial \theta^2} \log f(Y|\theta)\right) = -\frac{252}{\theta^2} + \frac{2\mathbb{E}(Y)}{\theta^3} = \frac{252}{\theta^2}.$$

4. (c) The Cramér-Rao inequality tells us that any unbiased estimator $\hat{\theta}$ of θ must satisfy

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{nI(\theta)} = \frac{\theta^2}{252n}.$$

Since

$$\mathrm{Var}(\hat{\theta}_1) = \frac{1}{252^2} \mathrm{Var}(\overline{Y}) = \frac{1}{252^2} \cdot \frac{252\theta^2}{n} = \frac{\theta^2}{252n},$$

we have found an unbiased estimator whose variance attains the lower bound of the Cramér-Rao inequality. Hence, $\hat{\theta}_1$ must be the MVUE of θ.