Stat 252 Winter 2016
Solutions to Assignment #2

1. (a) If $Y \sim \text{Unif}(0,\theta)$, then $f(y|\theta) = 1/\theta$ for $0 \le y \le \theta$ so that $\mathbb{E}(Y) = \theta/2$ and $\text{Var}(Y) = \theta^2/12$. Therefore,

$$\mathbb{E}(\hat{\theta}_1) = \mathbb{E}(2\overline{Y}) = 2\mathbb{E}(\overline{Y}) = 2 \cdot \frac{\theta}{2} = \theta$$

so that $\hat{\theta}_1$ is an unbiased estimator of θ. Moreover,

$$\text{MSE}(\hat{\theta}_1) = \text{Var}(\hat{\theta}_1) = \text{Var}(2\overline{Y}) = 4\,\text{Var}(\overline{Y}) = 4 \cdot \frac{\theta^2/12}{n} = \frac{\theta^2}{3n}.$$

1. (b) If $Y_{(1)} := \min\{Y_1, \dots, Y_n\}$ and $0 \le x \le \theta$, then

$$\mathbf{P}\left(Y_{(1)} > x\right) = [\mathbf{P}\left(Y_1 > x\right)]^n = \left[\frac{\theta - x}{\theta}\right]^n$$

so that

$$f_{Y_{(1)}}(x) = \frac{n}{\theta}\left[\frac{\theta - x}{\theta}\right]^{n-1} = n\theta^{-n}(\theta - x)^{n-1}, \quad 0 \le x \le \theta.$$

This implies

$$\mathbb{E}(Y_{(1)}) = \int_0^\theta x \cdot n\theta^{-n}(\theta - x)^{n-1}\,\mathrm{d}x = n\theta^{-n}\int_0^\theta x(\theta - x)^{n-1}\,\mathrm{d}x.$$

If we substitute $u = \theta - x$ so that

$$\mathbb{E}(Y_{(1)}) = n\theta^{-n}\int_0^\theta u^{n-1}(\theta - u)\,\mathrm{d}u = n\theta^{-n}\int_0^\theta \theta u^{n-1} - u^n\,\mathrm{d}u = n\theta^{-n}\left[\frac{\theta^{n+1}}{n} - \frac{\theta^{n+1}}{n+1}\right] = \frac{\theta}{n+1},$$

then

$$\mathbb{E}(\hat{\theta}_2) = \mathbb{E}((n+1)Y_{(1)}) = (n+1) \cdot \frac{\theta}{n+1} = \theta$$

so that $\hat{\theta}_2$ is an unbiased estimator of θ. Moreover,

$$\mathbb{E}(Y_{(1)}^2) = \int_0^\theta x^2 \cdot n\theta^{-n}(\theta - x)^{n-1}\,\mathrm{d}x = n\theta^{-n}\int_0^\theta x^2(\theta - x)^{n-1}\,\mathrm{d}x$$

so that if we substitute $u = \theta - x$, we find

$$\begin{aligned}
\mathbb{E}(Y_{(1)}^2) = n\theta^{-n}\int_0^\theta u^{n-1}(\theta - u)^2\,\mathrm{d}u &= n\theta^{-n}\int_0^\theta \theta^2 u^{n-1} - 2\theta u^n + u^{n+1}\,\mathrm{d}u \\
&= n\theta^{-n}\left[\frac{\theta^{n+2}}{n} - \frac{2\theta^{n+2}}{n+1} + \frac{\theta^{n+2}}{n+2}\right] \\
&= \left[1 - \frac{2n}{n+1} + \frac{n}{n+2}\right]\theta^2
\end{aligned}$$

and so

$$\text{Var}(Y_{(1)}) = \mathbb{E}(Y_{(1)}^2) - \left[\mathbb{E}(Y_{(1)})\right]^2 = \left[1 - \frac{2n}{n+1} + \frac{n}{n+2}\right]\theta^2 - \frac{\theta^2}{(n+1)^2}$$

Therefore,

$$\begin{aligned}\text{MSE}(\hat{\theta}_2) = \text{Var}(\hat{\theta}_2) = \text{Var}((n+1)Y_{(1)}) &= (n+1)^2\,\text{Var}(Y_{(1)}) \\ &= (n+1)^2\left(\left[1-\frac{2n}{n+1}+\frac{n}{n+2}\right]\theta^2-\frac{\theta^2}{(n+1)^2}\right) \\ &= \frac{n\theta^2}{n+2}.\end{aligned}$$

1. (c) If $Y_{(n)} := \max\{Y_1, \dots, Y_n\}$ and $0 \le x \le \theta$, then $\mathbf{P}\left(Y_{(n)} \le x\right) = [\mathbf{P}\left(Y_1 \le x\right)]^n = \theta^{-n}x^n$ so that $f_{Y_{(n)}}(x) = n\theta^{-n}x^{n-1}$ for $0 \le x \le \theta$. Therefore,

$$\mathbb{E}(Y_{(n)}) = \int_0^\theta x \cdot n\theta^{-n}x^{n-1}\,\mathrm{d}x = n\theta^{-n}\cdot\frac{\theta^{n+1}}{n+1} = \frac{n\theta}{n+1},$$

and so

$$\mathbb{E}(\hat{\theta}_3) = \frac{(n+1)}{n}\mathbb{E}(Y_{(n)}) = \theta$$

implying that $\hat{\theta}_3$ is an unbiased estimator of θ. Furthermore,

$$\mathbb{E}(Y_{(n)}^2) = \int_0^\theta x^2 \cdot n\theta^{-n}x^{n-1}\,\mathrm{d}x = n\theta^{-n}\cdot\frac{\theta^{n+2}}{n+2} = \frac{n\theta^2}{n+2}$$

so that

$$\text{Var}(Y_{(n)}) = \mathbb{E}(Y_{(n)}^2) - [\mathbb{E}(Y_{(n)})]^2 = \frac{n\theta^2}{n+2} - \frac{n^2\theta^2}{(n+1)^2}$$

and

$$\text{Var}(\hat{\theta}_3) = \frac{(n+1)^2}{n^2}\,\text{Var}(X) = \left(\frac{(n+1)^2}{n^2}\cdot\frac{n}{n+2}-1\right)\theta^2 = \frac{\theta^2}{n(n+2)}.$$

1. (d) We find

$$\text{Eff}(\hat{\theta}_1,\hat{\theta}_3) = \frac{\text{Var}(\hat{\theta}_3)}{\text{Var}(\hat{\theta}_1)} = \frac{\frac{\theta^2}{n(n+2)}}{\frac{\theta^2}{3n}} = \frac{3}{n+2} < 1$$

provided $n > 1$. Since $\text{Eff}(\hat{\theta}_1,\hat{\theta}_3) < 1$, we conclude that $\text{Var}(\hat{\theta}_3) < \text{Var}(\hat{\theta}_1)$ so that in this example $\hat{\theta}_3$ is preferred to $\hat{\theta}_1$.

1. (e) We find

$$\text{Eff}(\hat{\theta}_2,\hat{\theta}_3) = \frac{\text{Var}(\hat{\theta}_3)}{\text{Var}(\hat{\theta}_2)} = \frac{\frac{\theta^2}{n(n+2)}}{\frac{n\theta^2}{n+2}} = \frac{1}{n^2} < 1$$

provided $n > 1$. Since $\text{Eff}(\hat{\theta}_2,\hat{\theta}_3) < 1$, we conclude that $\text{Var}(\hat{\theta}_3) < \text{Var}(\hat{\theta}_2)$ so that in this example $\hat{\theta}_3$ is preferred to $\hat{\theta}_2$.

1. (f) As shown in class,

$$\text{Eff}(\hat{\theta}_1,\hat{\theta}_2) = \frac{\text{Var}(\hat{\theta}_2)}{\text{Var}(\hat{\theta}_1)} = \frac{\frac{n\theta^2}{n+2}}{\frac{\theta^2}{3n}} = \frac{3n^2}{n+2} > 1$$

provided $n > 1$. Since $\text{Eff}(\hat{\theta}_1,\hat{\theta}_2) > 1$, we conclude that $\text{Var}(\hat{\theta}_2) > \text{Var}(\hat{\theta}_1)$ so that in this example $\hat{\theta}_1$ is preferred to $\hat{\theta}_2$. By combining **(d)** and **(e)**, we conclude that $\hat{\theta}_3$ is preferred to either $\hat{\theta}_1$ or $\hat{\theta}_2$ provided $n > 1$. Note, however, that if $n = 1$, then $\hat{\theta}_1 = \hat{\theta}_2 = \hat{\theta}_3 = 2Y_1$.

2. (a) If $Y_{(1)} := \min\{Y_1, \ldots, Y_n\}$ and $x > 0$, then $\mathbf{P}\left(Y_{(1)} > x\right) = [\mathbf{P}\left(Y_1 > x\right)]^n = e^{-xn/\theta}$ implying that the density function for $Y_{(1)}$ is

$$f_{Y_{(1)}}(x) = \frac{n}{\theta} e^{-xn/\theta}, \quad x > 0.$$

That is, $Y_{(1)} \sim \text{Exp}(\theta/n)$ and so $\mathbb{E}(Y_{(1)}) = \theta/n$. Thus, if $\hat{\theta} := nY_{(1)}$, then $\mathbb{E}(\hat{\theta}) = n\mathbb{E}(Y_{(1)}) = n \cdot \theta/n = \theta$ so that $\hat{\theta}$ is an unbiased estimator of θ.

2. (b) Since $\hat{\theta}$ is an unbiased estimator of θ, we know $\text{MSE}(\hat{\theta}) = \text{Var}(\hat{\theta})$. However, since $Y_{(1)}$ is an exponential random variable with parameter θ/n, we know that $\text{Var}(Y_{(1)}) = \theta^2/n^2$ which implies

$$\text{MSE}(\hat{\theta}) = \text{Var}(\hat{\theta}) = \text{Var}(nY_{(1)}) = n^2 \, \text{Var}(Y_{(1)}) = n^2 \frac{\theta^2}{n^2} = \theta^2.$$

3. (a) If $\hat{\theta} := \max\{Y_1, \ldots, Y_n\}$ and $0 \le x \le \theta$, then

$$\mathbf{P}\left(\hat{\theta} \le x\right) = [\mathbf{P}\left(Y_1 \le x\right)]^n = \left[\int_0^x \alpha\theta^{-\alpha} y^{\alpha-1} \, \mathrm{d}y\right]^n = \theta^{-n\alpha} \, x^{n\alpha}$$

so that

$$f_{\hat{\theta}}(x) = n \, \alpha \, \theta^{-n\alpha} \, x^{n\alpha-1}, \quad 0 \le x \le \theta.$$

We easily calculate that

$$\mathbb{E}(\hat{\theta}) = \int_0^\theta x \cdot n \, \alpha \, \theta^{-n\alpha} \, x^{n\alpha-1} \, \mathrm{d}x = n \, \alpha \, \theta^{-n\alpha} \frac{\theta^{n\alpha+1}}{n\alpha+1} = \frac{n \, \alpha}{n \, \alpha + 1} \, \theta.$$

Thus, we conclude $\hat{\theta}$ is a biased estimator of θ.

3. (b) Clearly, the estimator

$$\hat{\theta}_1 := \frac{n \, \alpha + 1}{n \, \alpha} \, \hat{\theta} = \frac{n \, \alpha + 1}{n \, \alpha} \, \max\{Y_1, \ldots, Y_n\}$$

is an unbiased estimator of θ.

3. (c) We observe that since $\hat{\theta}_1$ is unbiased,

$$\text{MSE}(\hat{\theta}_1) = \text{Var}(\hat{\theta}_1) = \left(\frac{n \, \alpha + 1}{n \, \alpha}\right)^2 \text{Var}(\hat{\theta}).$$

In order to calculate $\text{Var}(\hat{\theta})$ we note that

$$\mathbb{E}(\hat{\theta}^2) = \int_0^\theta x^2 \cdot n \, \alpha \, \theta^{-n\alpha} \, x^{n\alpha-1} \, \mathrm{d}x = n \, \alpha \, \theta^{-n\alpha} \frac{\theta^{n\alpha+2}}{n\alpha+2} = \frac{n \, \alpha}{n \, \alpha + 2} \, \theta^2$$

and so

$$\text{Var}(\hat{\theta}) = \mathbb{E}(\hat{\theta}^2) - [\mathbb{E}(\hat{\theta})]^2 = \frac{n \, \alpha}{n \, \alpha + 2} \, \theta^2 - \left[\frac{n \, \alpha}{n \, \alpha + 1} \, \theta\right]^2 = \frac{n\alpha}{(n\alpha + 1)^2 (n\alpha + 2)} \, \theta^2.$$

Therefore,

$$\mathrm{MSE}(\hat{\theta}_1) = \left(\frac{n\,\alpha+1}{n\,\alpha}\right)^2 \frac{n\alpha}{(n\alpha+1)^2(n\alpha+2)}\,\theta^2 = \frac{1}{n\alpha(n\alpha+2)}\,\theta^2.$$

4. (a) If $X = \sqrt{Y_1 Y_2}$, then $\mathbb{E}(X) = \mathbb{E}\left(\sqrt{Y_1 Y_2}\right) = \mathbb{E}\left(\sqrt{Y_1}\right)\mathbb{E}\left(\sqrt{Y_2}\right) = \left[\mathbb{E}\left(\sqrt{Y_1}\right)\right]^2$ using the fact that Y_1, Y_2 are i.i.d. Since

$$\mathbb{E}\left(\sqrt{Y_1}\right) = \int_0^\infty \sqrt{y}\cdot\frac{1}{\theta}e^{-y/\theta}\,\mathrm{d}y = \sqrt{\theta}\int_0^\infty u^{1/2}e^{-u}\,\mathrm{d}u = \sqrt{\theta}\cdot\Gamma(3/2) = \sqrt{\theta}\cdot\frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi\theta}}{2}$$

using the facts from Stat 251 that $\Gamma(1+p) = p\Gamma(p)$ and $\Gamma(1/2) = \sqrt{\pi}$, we deduce that

$$\mathbb{E}(X) = \left(\frac{\sqrt{\pi\theta}}{2}\right)^2 = \frac{\pi}{4}\theta.$$

Thus,

$$\hat{\theta} := \frac{4}{\pi}\sqrt{Y_1 Y_2}$$

is an unbiased estimator of θ.

4. (b) If $W = \sqrt{Y_1 Y_2 Y_3 Y_4}$, then

$$\mathbb{E}(W) = \mathbb{E}\left(\sqrt{Y_1 Y_2 Y_3 Y_4}\right) = \left[\mathbb{E}\left(\sqrt{Y_1}\right)\right]^4 = \left[\frac{\sqrt{\pi\theta}}{2}\right]^4 = \frac{\pi^2}{16}\theta^2$$

as in **(a)** using the fact that Y_1, Y_2, Y_3, Y_4 are i.i.d. Thus,

$$\hat{\theta}_1 := \frac{16}{\pi^2}\sqrt{Y_1 Y_2 Y_3 Y_4}$$

is an unbiased estimator of θ^2.