Stat 252 Winter 2016
Assignment #6

This assignment is due at the beginning of class on Monday, March 14, 2016. Your solutions will be graded based on both correctness *and* exposition. In particular, neatness and grammar count. You must write out solutions using full sentences (including capital letters to start sentences and periods to end them) and no abbreviations. That is, symbols such as $\therefore$ and $\Rightarrow$ are forbidden; write out the full words *therefore* and *implies* in their place.

1. Recall that a discrete random variable Y is said to be Poisson with parameter $\theta > 0$ if the density (also called probability mass function) of Y is

$$f(y|\theta) = \frac{\theta^y e^{-y}}{y!}, \quad y = 0, 1, 2, \ldots.$$

Recall further that if Y is Poisson(θ), then $\mathbb{E}(Y) = \theta$ and $\mathrm{Var}(Y) = \theta$.

(a) It turns out that the same formula for the Fisher information can be used for discrete random variables. Show that if Y is a Poisson random variable with parameter θ, then $I(\theta) = \theta^{-1}$.

For parts (b), (c), (d), and (e) below, suppose that $Y_1, Y_2, \ldots, Y_n$ is a random sample from a Poisson(θ) population.

(b) Show that $\hat{\theta}_{\text{MOM}}$, the method of moments estimator of θ, is

$$\hat{\theta}_{\text{MOM}} = \overline{Y} = \frac{Y_1 + \cdots + Y_n}{n}.$$

(c) Show that $\hat{\theta}_{\text{MOM}}$ is an unbiased estimator of θ.

(d) Calculate $\mathrm{Var}(\hat{\theta}_{\text{MOM}})$.

(e) Explain why $\hat{\theta}_{\text{MOM}}$ must be the minimum variance unbiased estimator (MVUE) of θ.

2. Suppose that $Y_1, \ldots, Y_5$ are independent and identically distributed $\mathcal{N}(\mu, \sigma^2)$ random variables. As usual, let

$$\overline{Y} = \frac{1}{5}\sum_{i=1}^{5} Y_i \quad \text{and} \quad S^2 = \frac{1}{4}\sum_{i=1}^{5}(Y_i - \overline{Y})^2.$$

It is known that $\mu = 0$, but σ^2 is unknown. You want to test $H_0 : \sigma = 1$ against $H_A : \sigma > 1$ by rejecting H_0 when $S^2 > 1.945$. (*You may consider your calculations accurate to two decimal places when consulting the appropriate table(s).*)

(a) Verify that this test has significance level $\alpha = 0.10$.

(b) Using the test determined in (a), find the power of the test when $\sigma = 3.3$.

3. Suppose that $Y_1, \ldots, Y_n$ is a random sample from a population having common density function

$$f(y|\theta) = \frac{\theta^2}{y^3} \exp\{-\theta/y\}, \quad y > 0,$$

for some parameter $\theta > 0$. On Assignment #4, you showed that the maximum likelihood estimator of θ is

$$\hat{\theta}_{\text{MLE}} = \frac{2n}{\displaystyle\sum_{i=1}^{n} \frac{1}{Y_i}}.$$

(a) Use the Factorization Theorem to show that $\displaystyle\sum_{i=1}^{n} \frac{1}{Y_i}$ is a sufficient statistic for the estimation of θ.

(b) Explain why $\hat{\theta}_{\text{MLE}}$ must also be a sufficient statistic for the estimation of θ.

(c) Find the Fisher information $I(\theta)$ in a single observation from this density.

(d) Construct an approximate 90% confidence interval for θ based on the Fisher information and the maximum likelihood estimator. Choose your critical values to ensure that your confidence interval is symmetric (i.e., has equal upper and lower tail areas).

(e) Starting with your confidence interval for θ from (d), determine the rejection region for the test of the hypothesis $H_0 : \theta = \theta_0$ against $H_A : \theta \neq \theta_0$ at the (approximate) significance level $\alpha = 0.10$.

4. Assume that the outcome of an experiment is a single random variable Y. A 90% confidence interval for a parameter θ has the form $[Y - 2, Y + 3]$. From this, determine a rejection rule for testing $H_0 : \theta = 4$ against $H_A : \theta \neq 4$ at significance level 0.10.
(Hint: Use the confidence interval–hypothesis test duality.)

5. Let Y be a Uniform$[0, \theta]$ random variable. Consider testing $H_0 : \theta = 1$ against $H_A : \theta > 1$ by rejecting H_0 when $Y > c$.

(a) Find c so that this test has significance level 0.05.

(b) What is the power of the test in (a) (as a function of θ)?