Stat 252 Winter 2016
Assignment #2

This assignment is due at the beginning of class on Wednesday, January 20, 2016. Your solutions will be graded based on both correctness *and* exposition. In particular, neatness and grammar count. You must write out solutions using full sentences (including capital letters to start sentences and periods to end them) and no abbreviations. That is, symbols such as $\therefore$ and $\Rightarrow$ are forbidden; write out the full words *therefore* and *implies* in their place.

1. Suppose that $Y_1, \ldots, Y_n$ are a random sample from a Uniform$(0, \theta)$ distribution where $\theta > 0$ is a parameter. Consider the following three estimators of θ, namely

$$\hat{\theta}_1 = 2\overline{Y}, \qquad \hat{\theta}_2 = (n+1)\min\{Y_1, \ldots, Y_n\}, \qquad \text{and} \qquad \hat{\theta}_3 = \frac{(n+1)}{n}\max\{Y_1, \ldots, Y_n\}.$$

(a) Verify that $\hat{\theta}_1$ is an unbiased estimator of θ, and compute the mean-square error of $\hat{\theta}_1$.

(b) Verify that $\hat{\theta}_2$ is an unbiased estimator of θ, and compute the mean-square error of $\hat{\theta}_2$.

(c) Verify that $\hat{\theta}_3$ is an unbiased estimator of θ, and compute the mean-square error of $\hat{\theta}_3$.

(d) Compute Eff$(\hat{\theta}_1, \hat{\theta}_3)$. In this case, which of $\hat{\theta}_1$ and $\hat{\theta}_3$ is preferred for the estimation of θ?

(e) Compute Eff$(\hat{\theta}_2, \hat{\theta}_3)$. In this case, which of $\hat{\theta}_2$ and $\hat{\theta}_3$ is preferred for the estimation of θ?

(f) In class we computed Eff$(\hat{\theta}_1, \hat{\theta}_2)$. Combine this result along with your answers to **(d)** and **(e)** to decide which of $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$ is preferred for the estimation of θ.

2. Suppose that $Y_1, \ldots, Y_n$ are a random sample from a population with an exponential distribution whose density is

$$f(y|\theta) = \begin{cases} \frac{1}{\theta}e^{-y/\theta}, & y > 0, \\ 0, & y \le 0, \end{cases}$$

where $\theta > 0$ is a parameter.

(a) If $Y_{(1)} := \min\{Y_1, \ldots, Y_n\}$, show that $\hat{\theta} := nY_{(1)}$ is an unbiased estimator of θ.

(c) Compute MSE$(\hat{\theta})$.

3. Suppose that $Y_1, \ldots, Y_n$ are a random sample from a population having common density function

$$f(y|\theta) = \begin{cases} \alpha\theta^{-\alpha}y^{\alpha-1}, & 0 \le y \le \theta, \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha > 0$ is a known value and θ is a parameter. Consider the estimator $\hat{\theta} := \max\{Y_1, \ldots, Y_n\}$.

(a) Show that $\hat{\theta}$ is a biased estimator of θ.

(b) Find a multiple of $\hat{\theta}$ that is an unbiased estimator of θ. Call it $\hat{\theta}_1$.

(c) Compute MSE$(\hat{\theta}_1)$.

4. Suppose that Y_1, Y_2, Y_3, Y_4 are a random sample of size 4 from a population with an exponential distribution whose density is

$$f(y|\theta) = \begin{cases} \frac{1}{\theta} e^{-y/\theta}, & y > 0, \\ 0, & y \leq 0, \end{cases}$$

where $\theta > 0$ is a parameter.

(a) Let $X := \sqrt{Y_1 Y_2}$. Find a multiple of X that is an unbiased estimator of θ. Call it $\hat{\theta}$. (*Hint.* Since Y_1 and Y_2 are independent we know that $\mathbb{E}(\sqrt{Y_1 Y_2}) = \mathbb{E}(\sqrt{Y_1})\mathbb{E}(\sqrt{Y_2})$. Use your knowledge of the gamma function and the fact that $\Gamma(1/2) = \sqrt{\pi}$ to compute $\mathbb{E}(\sqrt{Y_1})$.)

(b) Let $W := \sqrt{Y_1 Y_2 Y_3 Y_4}$. Find a multiple of W that is an unbiased estimator of θ^2. Call it $\hat{\theta}_1$. Recall the hint from **(b)**.