Statistics 252 "Practice Midterm" (Solutions) – Winter 2007

1. Let $U = Y/\theta$ so that for $u > 0$,

$$P(U \le u) = P(Y \le \theta u) = \int_0^{\theta u} \frac{2y}{\theta^2} \exp\left\{-\frac{y^2}{\theta^2}\right\} dy = 1 - e^{-u^2}.$$

Thus, we must find a and b so that

$$\int_0^a 2ue^{-u^2}\, du = \frac{\alpha}{2} \quad \text{and} \quad \int_b^\infty 2ue^{-u^2}\, du = \frac{\alpha}{2}.$$

Computing the integrals we find $a = \sqrt{-\log(1-\alpha/2)}$ and $b = \sqrt{-\log(\alpha/2)}$. Hence,

$$\begin{aligned}
1 - \alpha = P(a \le U \le b) &= P\left(\sqrt{-\log(1-\alpha/2)} \le U \le \sqrt{-\log(\alpha/2)}\right) \\
&= P\left(\sqrt{-\log(1-\alpha/2)} \le \frac{Y}{\theta} \le \sqrt{-\log(\alpha/2)}\right) \\
&= P\left(\frac{Y}{\sqrt{-\log(\alpha/2)}} \le \theta \le \frac{Y}{\sqrt{-\log(1-\alpha/2)}}\right)
\end{aligned}$$

In other words,

$$\left[\frac{Y}{\sqrt{-\log(\alpha/2)}}, \frac{Y}{\sqrt{-\log(1-\alpha/2)}}\right]$$

is a confidence interval for θ with coverage probability $1 - \alpha$.

2. (a) The likelihood function is

$$L(\theta) = \prod_{i=1}^n f(y_i|\theta) = \theta^{2n} \left(\prod_{i=1}^n y_i\right)^{-3n} \exp\left\{-\theta \sum_{i=1}^n \frac{1}{y_i}\right\}.$$

(b) The log-likelihood function is

$$\ell(\theta) = 2n\log(\theta) - 3n\sum_{i=1}^n \log(y_i) - \theta \sum_{i=1}^n \frac{1}{y_i}.$$

Hence $\ell'(\theta) = 0$ implies

$$0 = \frac{2n}{\theta} - \sum_{i=1}^n \frac{1}{y_i}.$$

Since

$$\ell''(\theta) = -\frac{2n}{(\theta)^2} < 0,$$

we conclude that

$$\hat{\theta}_{\text{MLE}} = \frac{2n}{\displaystyle\sum_{i=1}^n \frac{1}{Y_i}}.$$

(c) If we let $u = \sum_{i=1}^{n} \frac{1}{y_i}$, then we can write

$$L(\theta) = g(u, \theta) \cdot h(y_1, \ldots, y_n)$$

where

$$h(y_1, \ldots, y_n) = \left(\prod_{i=1}^{n} y_i \right)^{-3n} \quad \text{and} \quad g(u, \theta) = \theta^{2n} \exp\{-\theta u\}$$

so by the Factorization Theorem we conclude that

$$\sum_{i=1}^{n} \frac{1}{Y_i}$$

is sufficient for θ.

(d) Recall that any one-to-one function of a sufficient statistic is also sufficient. Therefore, if we let

$$T(U) = \frac{2n}{U},$$

then since T is one-to-one, we find that

$$T\left(\sum_{i=1}^{n} \frac{1}{Y_i} \right) = \frac{2n}{\sum_{i=1}^{n} \frac{1}{Y_i}} = \hat{\theta}_{\text{MLE}}$$

is sufficient for θ.

(e) Since

$$\log f(y|\theta) = 2\log(\theta) - 3\log(y) - \frac{\theta}{y},$$

we find

$$\frac{\partial}{\partial \theta} \log f(y|\theta) = \frac{2}{\theta} - \frac{1}{y} \quad \text{and} \quad \frac{\partial^2}{\partial \theta^2} \log f(y|\theta) = \frac{-2}{\theta^2}.$$

Thus,

$$I(\theta) = -E\left(\frac{\partial^2}{\partial \theta^2} \log f(Y|\theta) \right) = \frac{2}{\theta^2}.$$

(f) Since an approximate $(1-\alpha)$ confidence interval for θ based on $\hat{\theta}_{\text{MLE}}$ is given by

$$\left[\hat{\theta}_{\text{MLE}} - z_{\alpha/2} \frac{1}{\sqrt{nI(\hat{\theta}_{\text{MLE}})}}, \hat{\theta}_{\text{MLE}} + z_{\alpha/2} \frac{1}{\sqrt{nI(\hat{\theta}_{\text{MLE}})}} \right],$$

we conclude that

$$\left[\frac{2n}{\sum_{i=1}^{n} \frac{1}{Y_i}} - 1.96 \frac{\sqrt{2n}}{\sum_{i=1}^{n} \frac{1}{Y_i}}, \frac{2n}{\sum_{i=1}^{n} \frac{1}{Y_i}} + 1.96 \frac{\sqrt{2n}}{\sum_{i=1}^{n} \frac{1}{Y_i}} \right],$$

is the required 95% confidence interval.

3. As a result of the confidence interval–hypothesis test duality, we know that the rejection region for the level 0.10 test of $H_0 : \theta = 4$ vs. $H_A : \theta \neq 4$ is

$$RR = \{4 \notin (Y - 2, Y + 3)\}.$$

That is, we reject H_0 in favour of H_A if $4 < Y - 2$ or $Y + 3 < 4$. In other words,

$$RR = \{Y < 1 \text{ or } Y > 6\}.$$

4. (a) By definition, the significance level α is the probability of a Type I error; that is, the probability under H_0 that H_0 is rejected, or

$$\alpha = P_{H_0}(\text{reject } H_0) = P(Y > c \,|\, \theta = 1).$$

If we assume that Y is Uniform$(0, 1)$, then $P(Y > c) = 1 - c$ so that in order to have a significance level 0.05 test, we need $c = 0.95$.

(b) By definition, the power of an hypothesis test is the probability under H_A that H_0 is rejected. That is,

$$\text{power} = P_{H_A}(\text{reject } H_0) = P(Y > 0.95 \,|\, \theta).$$

If we assume that Y is Uniform$(0, \theta)$, then

$$P(Y > 0.95) = \frac{\theta - 0.95}{\theta} = 1 - \frac{19}{20\,\theta}.$$