Statistics 252 "Practice Midterm" – Winter 2007

1. (*8 points*) Consider a random variable Y with density function

$$f_Y(y) = \frac{2y}{\theta^2} \exp\left\{-\frac{y^2}{\theta^2}\right\}, \quad y > 0,$$

where $\theta > 0$ is a parameter. Use the pivotal method to verify that if $0 < \alpha < 1$, then

$$\left[\frac{Y}{\sqrt{-\log(\alpha/2)}}, \frac{Y}{\sqrt{-\log(1-\alpha/2)}}\right]$$

is a confidence interval for θ with coverage probability $1 - \alpha$.

2. (*15 points*) Suppose that $Y_1, \ldots, Y_n$ are independent and identically distributed random variables with each Y_i having density function

$$f(y|\theta) = \frac{\theta^2}{y^3} \exp(-\theta/y), \quad y > 0$$

for some parameter $\theta > 0$.

(a) Compute the likelihood function $L(\theta)$ for this random sample.

(b) Show that the maximum likelihood estimator of θ is $\hat{\theta}_{\text{MLE}} = \dfrac{2n}{\sum_{i=1}^{n} \frac{1}{Y_i}}$.

(c) Show that $\sum_{i=1}^{n} \frac{1}{Y_i}$ is a sufficient statistic for the estimation of θ.

(d) Explain why $\hat{\theta}_{\text{MLE}}$ must also be a sufficient statistic for the estimation of θ.

(e) Find the Fisher information $I(\theta)$ in a single observation from this density.

(f) Construct an approximate 95% confidence interval for θ based on $\hat{\theta}_{\text{MLE}}$.

3. (*4 points*) Assume that the outcome of an experiment is a single random variable Y. A 90% confidence interval for a parameter θ has the form $(Y - 2, Y + 3)$. From this, determine a rejection rule for testing $H_0 : \theta = 4$ against $H_A : \theta \neq 4$ at significance level 0.10.
(Hint: Use the confidence interval–hypothesis test duality.)

4. (*8 points*) Let Y be a Uniform$(0, \theta)$ random variable. Consider testing $H_0 : \theta = 1$ against $H_A : \theta > 1$ by rejecting H_0 when $Y > c$.

(a) Find c so that this test has significance level 0.05.

(b) What is the power of the test in **(a)** (as a function of θ)?