Statistics 252 Midterm #2 – March 3, 2006

1. (*10 points*)

(a) Suppose that a random variable Y has density function $f_Y(y|\theta) = (\theta + 1)y^\theta$, $0 \le y \le 1$. Determine the Fisher information $I(\theta)$ for this random variable.

(b) Let $Y_1, \dots, Y_n$ be independent and identically distributed random variables each with the density $f_Y(y|\theta)$ given in **(a)**. Determine $\hat{\theta}_{\text{MOM}}$, the method of moments estimator of θ.

2. (*7 points*) Consider a random variable Y with density function $f_Y(y|\theta) = 2\theta^{-2}y$, $0 \le y \le \theta$, for some parameter $\theta > 0$. Use the pivotal method to verify that if $0 < \alpha < 1$, then

$$\left(\frac{Y}{\sqrt{1-\alpha/2}}, \frac{Y}{\sqrt{\alpha/2}} \right)$$

is a confidence interval for θ with coverage probability $1 - \alpha$.

3. (*10 points*) Recall that a discrete random variable Y is said to be Poisson with parameter $\theta > 0$ if the density (also called probability mass function) of Y is

$$f_Y(y|\theta) = \frac{\theta^y e^{-y}}{y!}, \quad y = 0, 1, 2, \dots.$$

Recall further that if Y is Poisson(θ), then $E(Y) = \theta$ and $\text{Var}(Y) = \theta$.

(a) It turns out that the same formula for the Fisher information can be used for discrete random variables. Show that if Y is a Poisson random variable with parameter θ, then $I(\theta) = \dfrac{1}{\theta}$.

For parts **(b)**, **(c)**, **(d)**, and **(e)** below, suppose that $Y_1, Y_2, \dots, Y_n$ are independent and identically distributed Poisson(θ) random variables.

(b) Show that $\hat{\theta}_{\text{MOM}}$, the method of moments estimator of θ, is $\hat{\theta}_{\text{MOM}} = \overline{Y} = \dfrac{Y_1 + \cdots + Y_n}{n}$.

(c) Show that $\hat{\theta}_{\text{MOM}}$ is an unbiased estimator of θ.

(d) Calculate $\text{Var}(\hat{\theta}_{\text{MOM}})$.

(e) Explain why $\hat{\theta}_{\text{MOM}}$ must be the minimum variance unbiased estimator (MVUE) of θ.

4. (*8 points*)

(a) Let $Y_1, Y_2, \dots, Y_n$ be independent and identically distributed random variables each with mean p and variance $p(1-p)$. Here p is an unknown parameter between 0 and 1. If

$$\hat{p} = \frac{Y_1 + \cdots + Y_n}{n}$$

then it is known that $\hat{p}$ is an unbiased estimator of p. Show that the maximum value of $\text{Var}(\hat{p})$, the variance of $\hat{p}$, occurs when $p = 1/2$.

(b) Suppose that $\hat{\theta}_1$ and $\hat{\theta}_2$ are both unbiased estimators of θ. Explain how $\text{Eff}(\hat{\theta}_1, \hat{\theta}_2)$, the relative efficiency of $\hat{\theta}_1$ to $\hat{\theta}_2$, can be used to decide which of these two estimator is preferable.