Stat 252 Winter 2007
Likelihood Ratio Test for a Normal Population

Suppose that $Y_1, \ldots, Y_n$ are independent and identically distributed $\mathcal{N}(\theta, 1)$ random variables where $-\infty < \theta < \infty$ is a parameter. Suppose that we are interested in testing the hypothesis $H_0 : \theta = \theta_0$ against $H_A : \theta \neq \theta_0$ using the generalized likelihood ratio test. Since the likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f_Y(y_i|\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(y_i - \theta)^2}{2}\right\} = (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(y_i - \theta)^2\right\}$$

and the maximum likelihood estimator is $\hat{\theta}_{\text{MLE}} = \overline{Y}$, we conclude that the likelihood ratio is

$$\begin{aligned}\Lambda(y_1, \ldots, y_n) = \frac{L(\theta_0)}{L(\hat{\theta}_{\text{MLE}})} &= \frac{(2\pi)^{-n/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(y_i - \theta_0)^2\right\}}{(2\pi)^{-n/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(y_i - \overline{y})^2\right\}} \\ &= \exp\left\{\frac{1}{2}\sum_{i=1}^{n}(y_i - \overline{y})^2 - \frac{1}{2}\sum_{i=1}^{n}(y_i - \theta_0)^2\right\}.\end{aligned}$$

Theorem. *If $Y_1, \ldots, Y_n$ are independent and identically distributed $\mathcal{N}(\theta, 1)$ random variables as above, and*

$$\Lambda = \Lambda(Y_1, \ldots, Y_n) = \exp\left\{\frac{1}{2}\sum_{i=1}^{n}(Y_i - \overline{Y})^2 - \frac{1}{2}\sum_{i=1}^{n}(Y_i - \theta_0)^2\right\},$$

then

$$-2\log\Lambda \sim \chi^2(1).$$

Proof. We begin by noting that

$$-2\log\Lambda = \sum_{i=1}^{n}(Y_i - \theta_0)^2 - \sum_{i=1}^{n}(Y_i - \overline{Y})^2 \qquad (*)$$

and so expanding the squares in $(*)$ gives

$$\begin{aligned}\left(\sum_{i=1}^{n} Y_i^2 - 2\theta_0\sum_{i=1}^{n} Y_i + n\theta_0^2\right) - \left(\sum_{i=1}^{n} Y_i^2 - 2\overline{Y}\sum_{i=1}^{n} Y_i + n\overline{Y}^2\right) &= n\theta_0^2 - 2n\theta_0\overline{Y} + n\overline{Y}^2 \\ &= n(\overline{Y} - \theta_0)^2.\end{aligned}$$

We now recall that the distribution of $\overline{Y}$ is

$$\overline{Y} \sim \mathcal{N}\left(\theta, \frac{1}{n}\right)$$

and so under the null hypothesis H_0, we conclude

$$\overline{Y} \sim \mathcal{N}\left(\theta_0, \frac{1}{n}\right).$$

Let

$$Z = \frac{\overline{Y} - \theta_0}{1/\sqrt{n}} = \sqrt{n}\left(\overline{Y} - \theta_0\right)$$

so that $Z \sim \mathcal{N}(0, 1)$ and

$$-2\log\Lambda = n(\overline{Y} - \theta_0)^2 = \left[\sqrt{n}(\overline{Y} - \theta_0)\right]^2 = Z^2.$$

Recalling that if $Z \sim \mathcal{N}(0, 1)$, then $Z^2 \sim \chi^2(1)$, we conclude that

$$-2\log\Lambda \sim \chi^2(1)$$

as required. □