Stat 252 Winter 2007
Generalized Likelihood Ratio Test

1. Suppose that $Y_1, \ldots, Y_n$ are independent and identically distributed random variables with each Y_i having density function

$$f(y|\theta) = \frac{\theta^2}{y^3} \exp\left(-\theta/y\right),$$

where $y > 0$ and $\theta > 0$. It is known that $\mathbb{E}(Y_i) = \theta$ and $\mathbb{E}\left(\frac{1}{Y_i}\right) = \frac{2}{\theta}$ for each $i = 1, \ldots, n$.

(a) Determine $\hat{\theta}_{\text{MOM}}$, the method of moments estimator of θ.

(b) Compute the likelihood function $L(\theta)$ for this random sample.

(c) Show that the maximum likelihood estimator of θ is $\hat{\theta}_{\text{MLE}} = \dfrac{2n}{\sum_{i=1}^{n} \frac{1}{Y_i}}$.

(d) Show that $\sum_{i=1}^{n} \frac{1}{Y_i}$ is a sufficient statistic for the estimation of θ.

(e) Explain why $\hat{\theta}_{\text{MLE}}$ must also be a sufficient statistic for the estimation of θ.

(f) Find the Fisher information $I(\theta)$ in a single observation from this density.

(g) Using the standard approximation for the distribution of a maximum likelihood estimator based on the Fisher information, construct an approximate 90% confidence interval for θ.

(h) Verify that the generalized likelihood ratio test for the test of the hypothesis $H_0 : \theta = \theta_0$ against $H_A : \theta \neq \theta_0$ has rejection region of the form

$$\left\{ \left(\sum_{i=1}^{n} \frac{1}{Y_i} \right)^{2n} \exp\left(-\theta_0 \sum_{i=1}^{n} \frac{1}{Y_i} \right) \leq C \right\}$$

for some constant C.

To answer **(i)** and **(j)** below, suppose that an observation of size $n = 8$ produces

$$\sum_{i=1}^{8} \frac{1}{Y_i} = 10.$$

(i) Based on your confidence interval constructed in **(g)** and on the above data, can you reject the hypothesis $H_0 : \theta = 1$ in favour of $H_A : \theta \neq 1$ at the significance level $\alpha = 0.10$?

(j) Based on your generalized likelihood ratio test constructed in **(h)** and on the above data, can you reject the hypothesis $H_0 : \theta = 1$ in favour of $H_A : \theta \neq 1$ at the significance level $\alpha = 0.10$?