Stat 252.01 Winter 2007
Assignment #9 Solutions

1. We find that $f_0(y) = e^{-y}$ for $y > 0$ and $f_A(y) = 2e^{-2y}$ for $y > 0$. Therefore, the likelihood ratio is

$$\Lambda(y) = \frac{f_0(y)}{f_A(y)} = \frac{e^{-y}}{2e^{-2y}} = \frac{1}{2}e^{y}$$

for $y > 0$, and so the rejection region is

$$RR = \{\Lambda(Y) < c\} = \left\{\frac{1}{2}e^{Y} < c\right\} = \{Y < \log(2c)\} = \{Y < c'\}$$

where $c' = \log(2c)$ is another constant. We now choose c (or, equivalently, c') so that this test has the desired significance level. Since

$$\alpha = P_{H_0}(\text{reject } H_0) = P_{\theta=1}(Y < c') = \int_0^{c'} f_0(y)\, dy = \int_0^{c'} e^{-y}\, dy = 1 - e^{-c'}$$

we conclude that $c' = -\log(1-\alpha)$ and so $RR = \{Y < -\log(1-\alpha)\}$.

2. (a) Recall that the generalized likelihood ratio test for the simple null hypothesis $H_0 : \theta = \theta_0$ against the composite alternative hypothesis $H_A : \theta \neq \theta_0$ has rejection region $\{\Lambda < c\}$ where

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta}_{\text{MLE}})}$$

is the generalized likelihood ratio and $L(\theta)$ is the likelihood function. In this instance,

$$L(\theta) = \theta^n \exp\left\{-\theta \sum_{i=1}^{n} y_i\right\}$$

so that

$$\Lambda = \frac{\theta_0^n \exp\left\{-\theta_0 \sum_{i=1}^{n} y_i\right\}}{\hat{\theta}_{\text{MLE}}^n \exp\left\{-\hat{\theta}_{\text{MLE}} \sum_{i=1}^{n} y_i\right\}} = \left(\frac{\theta_0}{1/\overline{Y}}\right)^n \exp\left\{-\theta_0 \sum y_i + 1/\overline{Y} \cdot \sum y_i\right\}$$

$$= \left(\theta_0 \overline{Y}\right)^n \exp\left\{n - n\theta_0\overline{Y}\right\} = e^n\, \theta_0^n\, \overline{Y}^{\,n} \exp\left\{-n\theta_0\overline{Y}\right\} = e^n\, \theta_0^n \left[\overline{Y} \exp\left\{-\theta_0\overline{Y}\right\}\right]^n.$$

Hence, we see that the rejection region $\{\Lambda < c\}$ can be expressed as

$$\left\{e^n\, \theta_0^n \left[\overline{Y} \exp\left\{-\theta_0\overline{Y}\right\}\right]^n < c\right\} = \left\{\overline{Y} \exp\left\{-\theta_0\overline{Y}\right\} < c^{1/n} e^{-1}\, \theta_0^{-1}\right\}$$
$$= \left\{\overline{Y} \exp\left\{-\theta_0\overline{Y}\right\} < C\right\}.$$

(To be explicit, the *suitable constant* is $C = c^{1/n} e^{-1}\, \theta_0^{-1}$.)

2. (b) We saw in class that $-2\log\Lambda \sim \chi^2(1)$ (approximately). This means that the generalized likelihood ratio test rejection region is $\{\Lambda < c\} = \{-2\log\Lambda > K\}$ where K is (yet another) constant. As we found above,

$$\Lambda = e^n\, \theta_0^n \left[\overline{Y} \exp\left\{-\theta_0\overline{Y}\right\}\right]^n$$

so that

$$-2\log\Lambda = -2n - 2n\log\theta_0 - 2n\log\overline{Y} + 2n\theta_0\overline{Y}.$$

Hence, to conduct the GLRT, we need to compare the observed value of $-2\log\Lambda$ with the appropriate chi-squared critical value which is $\chi^2_{0.10,1} = 2.70554$. Since

$$-2\cdot 10 - 2\cdot 10\log 1 - 2\cdot 10\cdot\log 1.25 + 2\cdot 10\cdot 1\cdot 1.25 \approx 2.76856$$

is the observed value of $-2\log\Lambda$, we reject H_0 at significance level 0.10. (Note, however, that since $\chi^2_{0.05,1} = 3.84146$, we fail to reject H_0 at significance level 0.05.)

3. (a) Recall that the generalized likelihood ratio test for the simple null hypothesis $H_0 : \theta = \theta_0$ against the composite alternative hypothesis $H_A : \theta \neq \theta_0$ has rejection region $\{\Lambda < c\}$ where

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta}_{\text{MLE}})}$$

is the generalized likelihood ratio and $L(\theta)$ is the likelihood function. In this instance,

$$L(\theta) = \theta^{2n}\left(\prod_{i=1}^{n} y_i\right)\exp\left\{-\theta\sum_{i=1}^{n} y_i\right\}$$

so that

$$\Lambda = \frac{\theta_0^{2n}\left(\prod_{i=1}^{n} y_i\right)\exp\left\{-\theta_0\sum_{i=1}^{n} y_i\right\}}{\hat{\theta}_{\text{MLE}}^{2n}\left(\prod_{i=1}^{n} y_i\right)\exp\left\{-\hat{\theta}_{\text{MLE}}\sum_{i=1}^{n} y_i\right\}} = \left(\frac{1}{2/\overline{Y}}\right)^{2n}\exp\left\{-\sum y_i + 2/\overline{Y}\cdot\sum y_i\right\}$$

$$= \left(\frac{\overline{Y}}{2}\right)^{2n}\exp\left\{2n - n\overline{Y}\right\}.$$

3. (b) We saw in class that $-2\log\Lambda \sim \chi^2(1)$ (approximately). This means that the generalized likelihood ratio test rejection region is $\{\Lambda < c\} = \{-2\log\Lambda > K\}$ where $K = -2\log c$ is (yet another) constant. (In fact, $K = \chi^2_{\alpha,1}$.) As we found above,

$$\Lambda = \left(\frac{\overline{Y}}{2}\right)^{2n}\exp\left\{2n - n\overline{Y}\right\}$$

so that

$$-2\log\Lambda = -4n\log\overline{Y} + 4n\log 2 - 4n + 2n\overline{Y}.$$

Hence, to conduct the GLRT, we need to compare the observed value of $-2\log\Lambda$ with the appropriate chi-squared critical value which is $\chi^2_{0.05,1} = 3.84146$. Since

$$-4\cdot 5\cdot\log 1 + 4\cdot 5\cdot\log 2 - 4\cdot 5 + 2\cdot 5\cdot 1 \approx 3.8629$$

is the observed value of $-2\log\Lambda$, we reject H_0 at significance level 0.05 (but just barely).