Stat 252.01 Winter 2007
Assignment #8 Solutions

1. As always, $\alpha = P_{H_0}(\text{reject } H_0)$ and $\beta = P_{H_A}(\text{accept } H_0)$. Since X has an Exponential(λ) distribution so that $f(x|\lambda) = \lambda^{-1}e^{-x/\lambda}$, $x > 0$, and since our rejection region is $\{X < c\}$, we find that

$$\alpha = P_{H_0}(\text{reject } H_0) = P(X < c|\lambda = 1) = \int_0^c e^{-x}\,dx = 1 - e^{-c}$$

and

$$\beta = P_{H_A}(\text{accept } H_0) = P(X > c|\lambda = 1/2) = \int_c^\infty 2e^{-2x}\,dx = e^{-2c}.$$

Thus, $\alpha = 1 - e^{-c}$ and $\beta = e^{-2c}$ which easily implies that

$$1 - \alpha = \sqrt{\beta}.$$

Rewrite this as $\alpha + \sqrt{\beta} = 1$ to illustrate the direct tradeoff between them: as α increases, β must decrease, and vice-versa.

2. If X has an Exponential(λ) distribution so that $f(x|\lambda) = \lambda^{-1}e^{-x/\lambda}$, $x > 0$, then the Fisher information is

$$I(\lambda) = \frac{1}{\lambda^2}$$

and the maximum likelihood estimator is

$$\hat{\lambda}_{\text{MLE}} = \overline{X}.$$

Hence, a significance level 0.1 test of $H_0 : \lambda = 1/5$ vs. $H_A : \lambda \neq 1/5$ has rejection region

$$RR = \left\{ \left| \sqrt{nI(\hat{\lambda}_{\text{MLE}})}\,(\hat{\lambda}_{\text{MLE}} - \lambda_0) \right| \geq z_{0.05} \right\}$$

or

$$RR = \left\{ \frac{\sqrt{n}}{\overline{X}} \left| \overline{X} - \frac{1}{5} \right| \geq 1.645 \right\}.$$

(10.10) Let μ denote the average hardness index. In order to test the manufacturer's claim, we want to test $H_0 : \mu \geq 64$ against $H_A : \mu < 64$. It is equivalent to test $H_0 : \mu = 64$ against $H_A : \mu < 64$. The test statistic is given by

$$Z = \frac{\overline{Y} - \mu_0}{\sigma/\sqrt{n}} = \frac{62 - 64}{8/\sqrt{50}} \approx -1.77.$$

In order to conduct this test at the significance level $\alpha = 0.01$, we find that the rejection region is

$$RR = \{Z < z_{0.01} = -2.326\}.$$

Since $Z = -1.77$ does not fall in the rejection region ($-1.77 > -2.326$), we do not reject H_0 in favour of H_A at the 0.01 level. Thus, we conclude that there is insufficient evidence to reject the manufacturer's claim.

(10.38) The rejection region is

$$\frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} < -z_\alpha$$

which is true if and only if

$$\hat{\theta} + z_\alpha \sigma_{\hat{\theta}} < \theta_0.$$

That is, H_0 will be rejected at the significance level α if and only if the $100(1 - \alpha)\%$ upper confidence bound for θ (namely, $\hat{\theta} + z_\alpha \sigma_{\hat{\theta}}$) is less than θ_0.

(10.50) A t-test can be used whenever one wants to conduct a hypothesis test of the population mean when the population is known to have a normal distribution with unknown variance. The t-test also works reasonably well for populations whose distribution is mound-shaped (and resembles the normal).

(10.73) Let σ denote the standard deviation of the accuracy of the precision instrument. In order to assess the precision, we want to test $H_0 : \sigma = 0.7$ against $H_A : \sigma > 0.7$. It is equivalent to test $H_0 : \sigma^2 = 0.49$ against $H_A : \sigma^2 > 0.49$. The sample variance is given by

$$\begin{aligned} s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \overline{y})^2 &= \frac{(353 - 352.5)^2 + (351 - 352.5)^2 + (351 - 352.5)^2 + (355 - 352.5)^2}{3} \\ &\approx 3.667 \end{aligned}$$

so that the test statistic is

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} \approx \frac{3 \cdot 3.667}{0.49} \approx 22.45.$$

(Recall that this is the distribution of the null hypothesis, i.e., assuming that $\sigma_0^2 = 0.49$.) Consulting Table 6 for the χ^2 density, we find $\chi^2_{0.005,3} = 12.8381$. Since $22.45 \gg 12.8381$, we see that the p-value must be smaller than 0.005.