Stat 252 Winter 2007
Assignment #5 Solutions

1. Textbook

(8.40) (a) By definition, if $0 < y < \theta$, then

$$F_Y(y) = \int_{-\infty}^{y} f_Y(u)\, du = \int_0^y \frac{2(\theta - u)}{\theta^2}\, du = \left. \frac{(2\theta u - u^2)}{\theta^2} \right|_0^y = \frac{(2\theta y - y^2)}{\theta^2} = \frac{2y}{\theta} - \frac{y^2}{\theta^2}.$$

That is,

$$F_Y(y) = \begin{cases} 0, & y \le 0, \\ \frac{2y}{\theta} - \frac{y^2}{\theta^2}, & 0 < y < 1, \\ 1, & y \ge \theta. \end{cases}$$

(b) If $U = Y/\theta$, then for $0 < u < 1$,

$$F_U(u) = P(U \le u) = P(Y \le u\theta) = \frac{2u\theta}{\theta} - \frac{u^2\theta^2}{\theta^2} = 2u - u^2.$$

Since the distribution of U does not depend on θ, this shows that $U = Y/\theta$ is a pivotal quantity.

(c) A 90% lower confidence limit for θ is therefore found by finding a such that $P(U > a) = 0.10$ for then we will have

$$P(U > a) = P\left(\frac{Y}{\theta} > a\right) = P\left(\theta < \frac{Y}{a}\right) = 0.10.$$

Solving

$$0.10 = P(U > a) = \int_a^1 f_u(u)\, du = \int_a^1 (2 - 2u)\, du = 1 - 2a + a^2$$

for a gives $a = 1 - \sqrt{0.10}$ (use the quadratic formula and reject the root for which $a > 1$) so that

$$P\left(\theta < \frac{Y}{1 - \sqrt{0.10}}\right) = 0.10 \quad \text{or, equivalently,} \quad P\left(\theta \ge \frac{Y}{1 - \sqrt{0.10}}\right) = 0.90.$$

(8.41) (a) A 90% upper confidence limit for θ is found by finding b such that $P(U < b) = 0.10$ for then we will have

$$P(U < b) = P\left(\frac{Y}{\theta} < b\right) = P\left(\theta > \frac{Y}{b}\right) = 0.10.$$

Solving

$$0.10 = P(U < b) = \int_0^b f_u(u)\, du = \int_0^b (2 - 2u)\, du = 1 - 2b + b^2$$

for b gives $b = 1 - \sqrt{0.90}$ (use the quadratic formula and reject the root for which $b > 1$) so that

$$P\left(\theta > \frac{Y}{1 - \sqrt{0.90}}\right) = 0.10 \quad \text{or, equivalently,} \quad P\left(\theta \le \frac{Y}{1 - \sqrt{0.90}}\right) = 0.90.$$

(b) We know from **(8.40) (c)** that

$$P\left(\theta < \frac{Y}{1-\sqrt{0.10}}\right) = 0.10$$

and we know from **(8.41) (a)** that

$$P\left(\theta > \frac{Y}{1-\sqrt{0.90}}\right) = 0.10.$$

Therefore,

$$P\left(\frac{Y}{1-\sqrt{0.90}} \le \theta \le \frac{Y}{1-\sqrt{0.10}}\right) = 0.80$$

so that

$$\left[\frac{Y}{1-\sqrt{0.90}}, \frac{Y}{1-\sqrt{0.10}}\right]$$

is an 80% confidence interval for θ.

(8.6) Recall that a Poisson(λ) random variable has mean λ and variance λ. This was also done in Stat 251.

(a) Since λ is the mean of a Poisson(λ) random variable, then a natural unbiased estimator for λ is

$$\hat{\lambda} = \overline{Y}.$$

(As you saw in problem **(8.4)**, there is NO unique unbiased estimator, so many other answers are possible.) It is a simple matter to compute that

$$\mathbb{E}(\hat{\lambda}) = \mathbb{E}(\overline{Y}) = \lambda \quad \text{and} \quad \mathrm{Var}(\hat{\lambda}) = \frac{\lambda}{n}.$$

We will need these in **(c)**.

(b) If $C = 3Y + Y^2$, then

$$\mathbb{E}(C) = \mathbb{E}(3Y) + \mathbb{E}(Y^2) = 3\mathbb{E}(Y) + (\mathrm{Var}(Y) + [\mathbb{E}(Y)]^2) = 3\lambda + (\lambda + \lambda^2) = 4\lambda + \lambda^2.$$

(c) This part is a little tricky. There is NO algorithm to solve it; instead you must THINK. Since $\mathbb{E}(C)$ depends on the *parameter* λ, we do not know its actual value. Therefore, we can *estimate* it. Suppose that $\theta = \mathbb{E}(C)$. Then, a *natural* estimator of $\theta = 4\lambda + \lambda^2$ is

$$\hat{\theta} = 4\hat{\lambda} + \hat{\lambda}^2,$$

where $\hat{\lambda} = \overline{Y}$ as in **(a)**. However, if we compute $\mathbb{E}(\hat{\lambda})$ we find

$$\mathbb{E}(\hat{\theta}) = \mathbb{E}(4\hat{\lambda}) + \mathbb{E}(\hat{\lambda}^2) = 4\mathbb{E}(\hat{\lambda}) + (\mathrm{Var}(\hat{\lambda}) + [\mathbb{E}(\hat{\lambda})]^2) = 4\lambda + \frac{\lambda}{n} + \lambda^2.$$

This does not equal θ, so that $\hat{\theta}$ is NOT unbiased. However, a little thought shows that if we *define*

$$\tilde{\theta} := 4\hat{\lambda} + \hat{\lambda}^2 - \frac{\hat{\lambda}}{n} = 4\overline{Y} + \overline{Y}^2 - \frac{\overline{Y}}{n},$$

then $\mathbb{E}(\tilde{\theta}) = 4\hat{\lambda} + \hat{\lambda}^2$ so that $\tilde{\theta}$ IS an unbiased estimator of $\theta = \mathbb{E}(C)$.

(8.8) If Y is a Uniform$(\theta, \theta+1)$ random variable, then its density is

$$f(y) = \begin{cases} 1, & \theta \le y \le \theta + 1, \\ 0, & \text{otherwise.} \end{cases}$$

It is a simple matter to compute

$$\mathbb{E}(Y) = \frac{2\theta + 1}{2} \quad \text{and} \quad \text{Var}(Y) = \frac{1}{12}.$$

(a) Hence,

$$\mathbb{E}(\overline{Y}) = \mathbb{E}\left(\frac{Y_1 + \cdots + Y_n}{n}\right) = \frac{\mathbb{E}(Y_1) + \cdots + \mathbb{E}(Y_n)}{n} = \frac{\frac{2\theta+1}{2} + \cdots + \frac{2\theta+1}{2}}{n} = \frac{2n\theta + n}{2n} = \theta + \frac{1}{2}.$$

We now find

$$B(\overline{Y}) = \mathbb{E}(\overline{Y}) - \theta = \left(\theta + \frac{1}{2}\right) - \theta = \frac{1}{2}.$$

(b) A little thought shows that our calculation in **(a)** immediately suggests a natural unbiased estimator of θ, namely

$$\hat{\theta} = \overline{Y} - \frac{1}{2}.$$

(c) We first compute that

$$\text{Var}(\overline{Y}) = \text{Var}\left(\frac{Y_1 + \cdots + Y_n}{n}\right) = \frac{\text{Var}(Y_1) + \cdots + \text{Var}(Y_n)}{n^2} = \frac{1/12 + \cdots + 1/12}{n^2} = \frac{1}{12n}.$$

As on page 367,

$$\text{MSE}(\overline{Y}) = \text{Var}(\overline{Y}) + [B(\overline{Y})]^2$$

so that

$$\text{MSE}(\overline{Y}) = \frac{1}{12n} + \left(\frac{1}{2}\right)^2 = \frac{3n+1}{12n}.$$

(8.9) **(a)** Let $\theta = \text{Var}(Y)$, and $\hat{\theta} = n(Y/n)(1 - Y/n)$. To prove $\hat{\theta}$ is unbiased, we must show that $\mathbb{E}(\hat{\theta}) \neq \theta$. Since

$$\mathbb{E}(\hat{\theta}) = \mathbb{E}(n(Y/n)(1 - Y/n)) = \mathbb{E}(Y) - \frac{1}{n}\mathbb{E}(Y^2),$$

and since Y is Binomial(n, p) so that $\mathbb{E}(Y) = np$, $\mathbb{E}(Y^2) = \text{Var}(Y) + [\mathbb{E}(Y)]^2 = np(1-p) + n^2p^2$, we conclude that

$$\mathbb{E}(\hat{\theta}) = np - \frac{np(1-p) + n^2p^2}{n} = (n-1)p(1-p).$$

(b) As an unbiased estimator, use

$$\frac{n}{n-1}\,\hat{\theta} = n\left(\frac{Y}{n-1}\right)\left(1 - \frac{Y}{n}\right).$$

(8.34) Let $\theta = V(Y) := \text{Var}(Y)$. If Y is a geometric random variable, then

$$\mathbb{E}(Y^2) = V(Y) + [\mathbb{E}(Y)]^2 = \frac{2}{p^2} - \frac{1}{p}.$$

Now a little thought shows that

$$\mathbb{E}\left(\frac{Y^2}{2} - \frac{Y}{2}\right) = \frac{1}{p^2} - \frac{1}{2p} - \frac{1}{2p} = \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2} = \theta.$$

Thus, choose

$$\hat{V}(Y) = \hat{\theta} = \frac{Y^2 - Y}{2}.$$

If Y is used to estimate $1/p$, then a two standard error bound on the error of estimation is given by

$$2\sqrt{\hat{V}(Y)} = 2\sqrt{\hat{\theta}} = 2\sqrt{\frac{Y^2 - Y}{2}}.$$

2. Textbook

(8.4) (a) Recall that if Y has the exponential density as given in the problem, then $\mathbb{E}(Y) = \theta$. In order to decide which estimators are unbiased, we simply compute $\mathbb{E}(\hat{\theta}_i)$ for each i. Four of these are easy:

$$\mathbb{E}(\hat{\theta}_1) = \mathbb{E}(Y_1) = \theta;$$

$$\mathbb{E}(\hat{\theta}_2) = \mathbb{E}\left(\frac{Y_1 + Y_2}{2}\right) = \frac{\mathbb{E}(Y_1) + \mathbb{E}(Y_2)}{2} = \frac{\theta + \theta}{2} = \theta;$$

$$\mathbb{E}(\hat{\theta}_3) = \mathbb{E}\left(\frac{Y_1 + 2Y_2}{3}\right) = \frac{\mathbb{E}(Y_1) + 2\mathbb{E}(Y_2)}{3} = \frac{\theta + 2\theta}{3} = \theta;$$

$$\mathbb{E}(\hat{\theta}_5) = \mathbb{E}(\overline{Y}) = \mathbb{E}\left(\frac{Y_1 + Y_2 + Y_3}{3}\right) = \frac{\mathbb{E}(Y_1) + \mathbb{E}(Y_2) + \mathbb{E}(Y_3)}{3} = \frac{\theta + \theta + \theta}{3} = \theta.$$

In order to compute $\mathbb{E}(\hat{\theta}_4) = \mathbb{E}(\min\{Y_1, Y_2, Y_3\})$ we need to do a bit of work, namely

$$\begin{aligned} P(\min\{Y_1, Y_2, Y_3\} > t) = P(Y_1 > t, Y_2 > t, Y_3 > t) &= P(Y_1 > t) \cdot P(Y_2 > t) \cdot P(Y_3 > t) \\ &= [P(Y_1 > t)]^3 \\ &= e^{-3t/\theta}. \end{aligned}$$

Thus, $f(t) = (3/\theta)e^{-3t/\theta}$, $t > 0$, which, as you will notice, is the density of an $\mathrm{Exp}(\theta/3)$ random variable. Thus,

$$\mathbb{E}(\hat{\theta}_4) = \mathbb{E}(\min\{Y_1, Y_2, Y_3\}) = \frac{\theta}{3}.$$

Hence, $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$, and $\hat{\theta}_5$ are unbiased, while $\hat{\theta}_4$ is biased.

(b) To decide which has the smallest variance, we simply compute. Recall that an $\mathrm{Exp}(\theta)$ random variable has variance θ^2. Thus,

$$\mathrm{Var}(\hat{\theta}_1) = \mathrm{Var}(Y_1) = \theta^2;$$

$$\mathrm{Var}(\hat{\theta}_2) = \mathrm{Var}\left(\frac{Y_1 + Y_2}{2}\right) = \frac{\mathrm{Var}(Y_1) + \mathrm{Var}(Y_2)}{4} = \frac{\theta^2 + \theta^2}{4} = \frac{\theta^2}{2};$$

$$\mathrm{Var}(\hat{\theta}_3) = \mathrm{Var}\left(\frac{Y_1 + 2Y_2}{3}\right) = \frac{\mathrm{Var}(Y_1) + 4\,\mathrm{Var}(Y_2)}{9} = \frac{\theta^2 + 4\theta^2}{9} = \frac{5\theta^2}{9};$$

$$\mathrm{Var}(\hat{\theta}_5) = \mathrm{Var}(\overline{Y}) = \mathrm{Var}\left(\frac{Y_1 + Y_2 + Y_3}{3}\right) = \frac{\mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) + \mathrm{Var}(Y_3)}{9} = \frac{\theta^2 + \theta^2 + \theta^2}{9} = \frac{\theta^2}{3};$$

and so $\hat{\theta}_5$ has the smallest variance. In fact, we will show later that it is the *minimum variance unbiased estimator*. That is, *no* other unbiased estimator of the mean will have smaller variance than $\overline{Y}$.

(9.7) If $\text{MSE}(\hat{\theta}_1) = \theta^2$, then $\text{Var}(\hat{\theta}_1) = \text{MSE}(\hat{\theta}_1) = \theta^2$ since $\hat{\theta}_1$ is unbiased. If $\hat{\theta}_2 = \overline{Y}$, then since the Y_i are exponential, we conclude $\mathbb{E}(\overline{Y}) = \theta$ and $\text{Var}(\overline{Y}) = \theta^2/n$. Thus,

$$\text{Eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\text{Var}(\hat{\theta}_2)}{\text{Var}(\hat{\theta}_1)} = \frac{1}{n}.$$

3. If $Y_1, \dots, Y_n$ are i.i.d. Uniform$(0, \theta)$, and $X := \max\{Y_1, \dots, Y_n\}$, then

$$P(X \le x) = P(Y_1 \le x) \cdots P(Y_n \le x) = \frac{x^n}{\theta^n}, \quad 0 \le x \le \theta.$$

It therefore follows that the density of X is

$$f_X(x) = \frac{nx^{n-1}}{\theta^n}, \quad 0 \le x \le \theta.$$

Hence,

$$\mathbb{E}(X) = \int_0^\theta x \cdot \frac{nx^{n-1}}{\theta^n}\, dx = \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1} = \frac{n\theta}{n+1},$$

and so if $\hat{\theta}_3 = \frac{(n+1)}{n} \max\{Y_1, \dots, Y_n\} = \frac{(n+1)}{n} X$, we find

$$\mathbb{E}(\hat{\theta}_3) = \frac{(n+1)\mathbb{E}(X)}{n} = \theta$$

so that $\hat{\theta}_3$ is an unbiased estimator of θ. Furthermore,

$$\mathbb{E}(X^2) = \int_0^\theta x^2 \cdot \frac{nx^{n-1}}{\theta^n}\, dx = \frac{n}{\theta^n} \cdot \frac{\theta^{n+2}}{n+2} = \frac{n\theta^2}{n+2}$$

so that

$$\text{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{n\theta^2}{n+2} - \frac{n^2\theta^2}{(n+1)^2}$$

and

$$\text{Var}(\hat{\theta}_3) = \frac{(n+1)^2}{n^2} \text{Var}(X) = \left(\frac{(n+1)^2}{n^2} \cdot \frac{n}{n+2} - 1 \right) \theta^2 = \frac{\theta^2}{n(n+2)}.$$

We now find

$$\text{Eff}(\hat{\theta}_1, \hat{\theta}_3) = \frac{\text{Var}(\hat{\theta}_3)}{\text{Var}(\hat{\theta}_1)} = \frac{\frac{\theta^2}{n(n+2)}}{\frac{\theta^2}{3n}} = \frac{3}{n+2} < 1$$

provided $n > 1$. Since $\text{Eff}(\hat{\theta}_1, \hat{\theta}_3) < 1$, we conclude that $\text{Var}(\hat{\theta}_3) < \text{Var}(\hat{\theta}_1)$ so that in this example $\hat{\theta}_3 := \frac{(n+1)}{n} \max\{Y_1, \dots, Y_n\}$ is preferred to $\hat{\theta}_1 := 2\overline{Y}$.

4. Since

$$\log f(y|\theta) = 2\log(\theta) - \theta^2 y,$$

we find

$$\frac{\partial^2}{\partial\theta^2} \log f(y|\theta) = \frac{-2}{\theta^2} - 2y.$$

Thus,

$$I(\theta) = -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\log f(Y|\theta)\right) = \frac{2}{\theta^2} + 2\mathbb{E}(Y) = \frac{4}{\theta^2}$$

since $\mathbb{E}(Y) = \theta^{-2}$. (This is because $Y \sim \text{Exp}(\theta^{-2})$.)

5. Standard Normal Handout

(1.) Observe that

$$1 - \Phi(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx - \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \int_{z}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx.$$

Let $u = -x$ so that $du = -dx$ and

$$\int_{z}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = -\int_{-z}^{-\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du = \int_{-\infty}^{-z} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du = \Phi(-z).$$

That is, $1 - \Phi(z) = \Phi(-z)$ as required.

(2.) If $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(u-\mu)^2}{2\sigma^2}}\,du.$$

Let $z = \frac{u-\mu}{\sigma}$ so that $\sigma\,dz = du$ and

$$\int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(u-\mu)^2}{2\sigma^2}}\,du = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

as required.

(3.) Consider

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2}\,dx$$

and let $u = \sqrt{2}\,x$ so that $du = \sqrt{2}\,dx$ and

$$\text{erf}(z) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\sqrt{2}\,z} e^{-\frac{u^2}{2}}\,du = 2 \cdot \frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{2}\,z} e^{-\frac{u^2}{2}}\,du. \tag{†}$$

However,

$$\frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{2}\,z} e^{-\frac{u^2}{2}}\,du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{2}\,z} e^{-\frac{u^2}{2}}\,du - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{u^2}{2}}\,du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{2}\,z} e^{-\frac{u^2}{2}}\,du - \frac{1}{2}$$
$$= \Phi(\sqrt{2}\,z) - \frac{1}{2}$$

so that (†) implies

$$\text{erf}(z) = 2\,\Phi(\sqrt{2}\,z) - 1$$

as required.

(5.) By definition,

$$\mathbb{E}(\ (ae^{bZ} - K)^+\ ) = \int_{-\infty}^{\infty} (ae^{bz} - K)^+ \phi(z)\, dz.$$

Observe, however, that $(ae^{bz} - K)^+ := \max\{ae^{bz} - K, 0\}$ which implies that the integral above is non-zero provided that $ae^{bz} - K \geq 0$ or $z \geq \frac{1}{b}\log(K/a)$. Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} (ae^{bz} - K)^+ \phi(z)\, dz &= \int_{\frac{1}{b}\log(K/a)}^{\infty} (ae^{bz} - K)\phi(z)\, dz \\ &= a\int_{\frac{1}{b}\log(K/a)}^{\infty} e^{bz}\phi(z)\, dz - K\int_{\frac{1}{b}\log(K/a)}^{\infty} \phi(z)\, dz.\end{aligned}$$

We now consider separately these last two integrals. We first find

$$\int_{\frac{1}{b}\log(K/a)}^{\infty} e^{bz}\phi(z)\, dz = \frac{1}{\sqrt{2\pi}}\int_{\frac{1}{b}\log(K/a)}^{\infty} e^{bz}e^{-\frac{z^2}{2}}\, dz$$

and so completing the square gives

$$\frac{1}{\sqrt{2\pi}}\int_{\frac{1}{b}\log(K/a)}^{\infty} e^{bz}e^{-\frac{z^2}{2}}\, dz = \frac{e^{b^2/2}}{\sqrt{2\pi}}\int_{\frac{1}{b}\log(K/a)}^{\infty} \exp\left\{-\frac{(z-b)^2}{2}\right\}\, dz.$$

Letting $u = z - b$ so that $du = dz$ we find

$$\frac{e^{b^2/2}}{\sqrt{2\pi}}\int_{\frac{1}{b}\log(K/a)}^{\infty} \exp\left\{-\frac{(z-b)^2}{2}\right\}\, dz = \frac{e^{b^2/2}}{\sqrt{2\pi}}\int_{\frac{1}{b}\log(K/a)-b}^{\infty} e^{-\frac{u^2}{2}}\, dz = e^{b^2/2}\ \Phi\left(b + \frac{1}{b}\log\frac{a}{K}\right).$$

Next we find

$$K\int_{\frac{1}{b}\log(K/a)}^{\infty} \phi(z)\, dz = K\left[1 - \Phi\left(\frac{1}{b}\log(K/a)\right)\right] = K\Phi\left(-\frac{1}{b}\log(K/a)\right) = K\Phi\left(\frac{1}{b}\log\frac{a}{K}\right).$$

Combining everything we conclude that

$$\mathbb{E}(\ (ae^{bZ} - K)^+\ ) = ae^{b^2/2}\ \Phi\left(b + \frac{1}{b}\log\frac{a}{K}\right) - K\Phi\left(\frac{1}{b}\log\frac{a}{K}\right).$$

6. Incomplete Gamma Function Handout

(1.) By definition,

$$\Gamma(a; y) := \int_0^y x^{a-1}\, e^{-x}\, dx.$$

If $u = x^{a-1}$ and $dv = e^{-x}\, dx$ so that $du = (a-1)x^{a-2}\, dx$ and $v = -e^{-x}$, then integration by parts gives

$$\int_0^y x^{a-1}\, e^{-x}\, dx = -x^{a-1}e^{-x}\Big|_0^y + (a-1)\int_0^y x^{a-2}e^{-x}\, dx = -y^{a-1}e^{-y} + (a-1)\int_0^y x^{a-2}e^{-x}\, dx.$$

That is,

$$\Gamma(a; y) = -y^{a-1}e^{-y} + (a-1)\Gamma(a-1; y)$$

as required.

Since $\Gamma(a;y) = (a-1)\Gamma(a-1;y) - e^{-y}y^{a-1}$ we can solve for $\Gamma(a-1;y)$ to conclude

$$\Gamma(a-1;y) = \frac{1}{a-1}\left[\Gamma(a;y) + e^{-y}y^{a-1}\right].$$

Replacing $a-1$ by a gives

$$\Gamma(a;y) = \frac{1}{a}\left[\Gamma(a+1;y) + e^{-y}y^{a}\right]$$

as required.

(2.) By definition,

$$G(a;y) := \int_y^\infty x^{a-1}\,e^{-x}\,dx,$$

and so we have

$$\Gamma(a;y) + G(a;y) = \Gamma(a).$$

Since $\Gamma(a;y) = (a-1)\Gamma(a-1;y) - e^{-y}y^{a-1}$ we conclude that

$$\Gamma(a) - G(a;y) = (a-1)[\Gamma(a-1) - G(a-1;y)] - e^{-y}y^{a-1}$$

and so

$$\begin{aligned} G(a;y) &= \Gamma(a) - (a-1)\Gamma(a-1) + (a-1)G(a-1;y) + e^{-y}y^{a-1} \\ &= (a-1)G(a-1;y) + e^{-y}y^{a-1} \end{aligned}$$

using the fact that $(a-1)\Gamma(a-1) = \Gamma(a)$.

Solving $G(a;y) = (a-1)G(a-1;y) + e^{-y}y^{a-1}$ for $G(a-1;y)$ we conclude

$$G(a-1;y) = \frac{1}{a-1}\left[G(a;y) - e^{-y}y^{a-1}\right].$$

Replacing $a-1$ by a gives

$$G(a;y) = \frac{1}{a}\left[G(a+1;y) - e^{-y}y^{a}\right]$$

as required.