Stat 252 Winter 2007
Assignment #3 Solutions

2. (a) Since $Y_1 \sim \mathcal{U}(0, \theta)$, we have

$$f_Y(y) = \begin{cases} 1/\theta, & 0 \le y \le \theta, \\ 0, & \text{otherwise.} \end{cases}$$

(b) If $f(t)$ denotes the density of $\hat{\theta} = \min(Y_1, \ldots, Y_n)$, then $f(t) = F'(t)$, where

$$F(t) = P(\hat{\theta} \le t) = 1 - P(\hat{\theta} > t) = 1 - P(Y_1 > t, \ldots, Y_n > t) = 1 - [P(Y_1 > t)]^n.$$

Note that in the last step we have used the fact that Y_i are i.i.d. Next, for $0 \le t \le \theta$, we compute

$$P(Y_1 > t) = \int_t^\infty f_Y(y)\, dy = \int_t^\theta \frac{1}{\theta}\, dy = \frac{\theta - t}{\theta}.$$

Thus, we conclude

$$f(t) = \frac{d}{dt}\left(1 - \left[\frac{\theta - t}{\theta}\right]^n\right) = n\theta^{-n}(\theta - t)^{n-1}, \quad 0 \le t \le \theta.$$

(c) By definition,

$$E(\hat{\theta}) = \int_{-\infty}^{\infty} t f(t)\, dt = \int_0^\theta n\theta^{-n} t(\theta - t)^{n-1}\, dt.$$

This last integral is solved with a simple substitution. Let $u = \theta - t$ so that $du = -dt$. Thus,

$$\begin{aligned} \int_0^\theta n\theta^{-n} t(\theta - t)^{n-1}\, dt = -n\theta^{-n} \int_\theta^0 (\theta - u)u^{n-1}\, du &= n\theta^{-n} \int_0^\theta \theta u^{n-1} - u^n\, du \\ &= n\theta^{-n}\left(\theta \cdot \frac{\theta^n}{n} - \frac{\theta^{n+1}}{n+1}\right) \\ &= \frac{\theta}{n+1}. \end{aligned}$$

(d) From **(c)**, we clearly see that $\hat{\theta}$ is NOT an unbiased estimator of θ. However,

$$\tilde{\theta} = (n+1)\min(Y_1, \ldots, Y_n)$$

IS an unbiased estimator of θ. (You should check that $2\overline{Y}$ is also an unbiased estimator of θ. Why?)