Statistics 252 Winter 2006 Midterm #3 – Solutions

1. (a) By definition, the significance level α is the probability of a Type I error; that is, the probability under H_0 that H_0 is rejected. Hence, since $\frac{4S^2}{\sigma^2} \sim \chi^2_4$,

$$\alpha = P_{H_0}(\text{reject } H_0) = P(S^2 > 1.945 | \sigma^2 = 1) = P\left(\frac{4S^2}{1} > \frac{4 \cdot 1.945}{1}\right)$$
$$= P(\chi^2 > 7.78) = 0.10,$$

where $\chi^2 \sim \chi^2_4$. (The last step follows from Table 6.) Hence, we see that the hypothesis test does, in fact, have significance level $\alpha = 0.10$.

(b) By definition, the power of an hypothesis test is the probability under H_A that H_0 is rejected. Hence, when $\sigma = 3.3$, we find

$$\text{power} = P_{H_A}(\text{reject } H_0) = P(S^2 > 1.945 | \sigma^2 = 3.3^2) = P\left(\frac{4S^2}{3.3^2} > \frac{4 \cdot 1.945}{3.3^2}\right)$$
$$= P(\chi^2 > 0.71) = 0.95,$$

where $\chi^2 \sim \chi^2_4$. (The last step follows from Table 6.) Hence, the power of this test when $\sigma = 3.3$ is 0.95.

2. (a) The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \theta^{2n}\left(\prod_{i=1}^{n} y_i\right)^{-3n} \exp\left\{-\theta \sum_{i=1}^{n} \frac{1}{y_i}\right\}.$$

(b) The log-likelihood function is

$$\ell(\theta) = 2n\log(\theta) - 3n\sum_{i=1}^{n}\log(y_i) - \theta\sum_{i=1}^{n}\frac{1}{y_i}.$$

Hence $\ell'(\theta) = 0$ implies

$$0 = \frac{2n}{\theta} - \sum_{i=1}^{n}\frac{1}{y_i}.$$

Since

$$\ell''(\theta) = -\frac{2n}{(\theta)^2} < 0,$$

we conclude that

$$\hat{\theta}_{\text{MLE}} = \frac{2n}{\sum_{i=1}^{n}\frac{1}{Y_i}}.$$

(c) If we let $u = \sum_{i=1}^{n}\frac{1}{y_i}$, then we can write

$$L(\theta) = g(u, \theta) \cdot h(y_1, \ldots, y_n)$$

where

$$h(y_1, \ldots, y_n) = \left(\prod_{i=1}^{n} y_i\right)^{-3n} \quad \text{and} \quad g(u, \theta) = \theta^{2n} \exp\{-\theta u\}$$

so by the Factorization Theorem we conclude that

$$\sum_{i=1}^{n} \frac{1}{Y_i}$$

is sufficient for θ.

(d) Recall that any one-to-one function of a sufficient statistic is also sufficient. Therefore, if we let

$$T(U) = \frac{2n}{U},$$

then since T is one-to-one, we find that

$$T\left(\sum_{i=1}^{n} \frac{1}{Y_i}\right) = \frac{2n}{\sum_{i=1}^{n} \frac{1}{Y_i}} = \hat{\theta}_{\text{MLE}}$$

is sufficient for θ.

(e) Since

$$\log f(y|\theta) = 2\log(\theta) - 3\log(y) - \frac{\theta}{y},$$

we find

$$\frac{\partial}{\partial\theta} \log f(y|\theta) = \frac{2}{\theta} - \frac{1}{y} \quad \text{and} \quad \frac{\partial^2}{\partial\theta^2} \log f(y|\theta) = \frac{-2}{\theta^2}.$$

Thus,

$$I(\theta) = -E\left(\frac{\partial^2}{\partial\theta^2} \log f(Y|\theta)\right) = \frac{2}{\theta^2}.$$

(f) The rejection region of a significance level 0.05 test of $H_0 : \theta = \theta_0$ vs. $H_A : \theta \neq \theta_0$ based on the Fisher information and the MLE is

$$\left\{\sqrt{nI(\hat{\theta}_{\text{MLE}})}\ \left|\hat{\theta}_{\text{MLE}} - \theta_0\right| \geq z_{0.025}\right\}$$

or

$$\left\{\left|\sqrt{2n} - \frac{\theta_0}{\sqrt{2n}} \sum_{i=1}^{n} \frac{1}{Y_i}\right| \geq 1.96\right\}.$$

3. As a result of the confidence interval–hypothesis test duality, we know that the rejection region for the level 0.10 test of $H_0 : \theta = 4$ vs. $H_A : \theta \neq 4$ is

$$RR = \{4 \notin (Y - 2, Y + 3)\}.$$

That is, we reject H_0 in favour of H_A if $4 < Y - 2$ or $Y + 3 < 4$. In other words,

$$RR = \{Y < 1 \text{ or } Y > 6\}.$$

4. (a) By definition, the significance level α is the probability of a Type I error; that is, the probability under H_0 that H_0 is rejected, or

$$\alpha = P_{H_0}(\text{reject } H_0) = P(Y > c \,|\, \theta = 1).$$

If we assume that Y is Uniform$(0,1)$, then $P(Y > c) = 1 - c$ so that in order to have a significance level 0.05 test, we need $c = 0.95$.

(b) By definition, the power of an hypothesis test is the probability under H_A that H_0 is rejected. That is,

$$\text{power} = P_{H_A}(\text{reject } H_0) = P(Y > 0.95 \,|\, \theta).$$

If we assume that Y is Uniform$(0, \theta)$, then

$$P(Y > 0.95) = \frac{\theta - 0.95}{\theta} = 1 - \frac{19}{20\,\theta}.$$