Statistics 252 Winter 2006 Midterm #1 – Solutions

1. (a) Since Y_1, Y_2, Y_3 are independent and identically distributed we find

$$E(\overline{Y}) = E\left(\frac{Y_1+Y_2+Y_3}{3}\right) = \frac{E(Y_1)+E(Y_2)+E(Y_3)}{3} = \frac{3E(Y_1)}{3} = E(Y_1).$$

We also compute that

$$E(Y_1) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = \int_0^\theta 2\theta^{-2}y^2\, dy = \frac{2}{3}\theta^{-2}y^3\Big|_0^\theta = \frac{2}{3}\theta^{-2}\theta^3 = \frac{2}{3}\theta.$$

Therefore,

$$E(\hat{\theta}_1) = E\left(\frac{3}{2}\overline{Y}\right) = \frac{3}{2}E(\overline{Y}) = \frac{3}{2}E(Y_1) = \frac{3}{2}\cdot\frac{2}{3}\theta = \theta.$$

Hence, $\hat{\theta}_1$ is an unbiased estimator of θ.

1. (b) Since Y_1, Y_2, Y_3 are independent and identically distributed we find

$$P(\max\{Y_1,Y_2,Y_3\} \le t) = P(Y_1 \le t, Y_2 \le t, Y_3 \le t) = [\,P(Y_1 \le t)\,]^3.$$

Hence, for $0 \le t \le \theta$,

$$P(Y_1 \le t) = \int_0^t 2\theta^{-2}y\, dy = \theta^{-2}t^2$$

so that

$$P(\max\{Y_1,Y_2,Y_3\} \le t) = \theta^{-6}t^6.$$

Therefore, the density function of $\max\{Y_1,Y_2,Y_3\}$ is

$$f(t) = 6\theta^{-6}t^5, \quad 0 \le t \le \theta$$

so that

$$E(\max\{Y_1,Y_2,Y_3\}) = \int_{-\infty}^{\infty} t\, f(t)\, dt = \int_0^\theta 6\theta^{-6}t^6\, dt = \frac{6}{7}\theta.$$

Finally, we conclude that

$$E(\hat{\theta}_2) = \frac{7}{6}E(\max\{Y_1,Y_2,Y_3\}) = \frac{7}{6}\cdot\frac{6}{7}\theta = \theta$$

which shows that $\hat{\theta}_2$ is an unbiased estimator of θ.

1. (c) When comparing unbiased estimators, we prefer the one with the smallest variance. We begin with $\hat{\theta}_1$, and find that

$$\mathrm{Var}(\overline{Y}) = \mathrm{Var}\left(\frac{Y_1+Y_2+Y_3}{3}\right) = \frac{\mathrm{Var}(Y_1)+\mathrm{Var}(Y_2)+\mathrm{Var}(Y_3)}{3^2} = \frac{3\,\mathrm{Var}(Y_1)}{9} = \frac{\mathrm{Var}(Y_1)}{3}.$$

Now,

$$E(Y_1^2) = \int_0^\theta 2\theta^{-2}y^3\, dy = \frac{2}{4}\theta^{-2}\theta^4 = \frac{1}{2}\theta^2.$$

so that

$$\operatorname{Var}(Y_1) = E(Y_1^2) - [E(Y_1)]^2 = \frac{1}{2}\theta^2 - \left(\frac{2}{3}\theta\right)^2 = \frac{1}{18}\theta^2.$$

Finally, we conclude

$$\operatorname{Var}(\hat{\theta}_1) = \operatorname{Var}\left(\frac{3}{2}\overline{Y}\right) = \frac{3^2}{2^2}\operatorname{Var}(\overline{Y}) = \frac{3^2}{2^2}\frac{\operatorname{Var}(Y_1)}{3} = \frac{3}{4}\cdot\frac{1}{18}\theta^2 = \frac{1}{24}\theta^2.$$

As for $\hat{\theta}_2$, we have from **(b)** that the density function of $\max\{Y_1, Y_2, Y_3\}$ is $f(t) = 6\theta^{-6}t^5$, $0 \le t \le \theta$. Therefore,

$$E(\hat{\theta}_2^2) = E\left(\left[\frac{7}{6}\max\{Y_1, Y_2, Y_3\}\right]^2\right) = \frac{7^2}{6^2}\int_0^\theta 6\theta^{-6}t^7\,dt = \frac{7^2}{6^2}\cdot\frac{6}{8}\theta^{-6}\theta^8 = \frac{7^2}{6\cdot 8}\theta^2 = \frac{49}{48}\theta^2.$$

Hence we conclude

$$\operatorname{Var}(\hat{\theta}_2^2) = E(\hat{\theta}_2^2) - [E(\hat{\theta}_2)]^2 = \frac{49}{48}\theta^2 - \theta^2 = \frac{1}{48}\theta^2.$$

In conclusion, we prefer $\hat{\theta}_2^2$ because of the two estimators given, it has the smaller variance.

2. Suppose that Y_1 and Y_2 denote the amount of time that the first and second customer, respectively, spend with the clerk. Then the probability we are required to compute is $P(Y_1 + Y_2 > 24)$. Therefore,

$$P(Y_1 + Y_2 > 24) = P\left(\frac{Y_1 + Y_2}{2} > \frac{24}{2}\right) = P(\overline{Y} > 12).$$

Since Y_1 and Y_2 are independent $\mathcal{N}(10, 8)$ random variables, we know that $\overline{Y} \sim \mathcal{N}(10, \frac{8}{2})$. Therefore, normalizing gives

$$P(\overline{Y} > 12) = P\left(\frac{\overline{Y} - 10}{\sqrt{4}} > \frac{12 - 10}{\sqrt{4}}\right) = P(Z > 1)$$

where $Z \sim \mathcal{N}(0, 1)$. From Table 4, we find $P(Z > 1) \approx 0.1587$ so that we conclude $P(Y_1 + Y_2 > 24) \approx 0.1587$.

3. (a) Since $Y_1, \ldots, Y_n$ are independent and identically distributed we find

$$E(\overline{Y}) = E\left(\frac{Y_1 + \cdots + Y_n}{n}\right) = \frac{E(Y_1) + \cdots + E(Y_n)}{n} = \frac{nE(Y_n)}{n} = E(Y_1) = \lambda$$

recalling that a Poisson random variable has expected value λ.

3. (b) Following the hint, we expand the square and find

$$S^2 = \frac{1}{n-1}\sum_{i=1}^n (Y_i - \overline{Y})^2 = \frac{1}{n-1}\left(\sum_{i=1}^n Y_i^2 - n\overline{Y}^2\right).$$

Therefore,

$$E(S^2) = \frac{1}{n-1}\left(\sum_{i=1}^n E(Y_i^2) - nE(\overline{Y}^2)\right).$$

In order to compute $E(Y_i^2)$ and $E(\overline{Y})^2$ we remember that

- $E(Y_i^2) = \mathrm{Var}(Y_i) + [E(Y_i)]^2$, and
- $E(\overline{Y}^2) = \mathrm{Var}(\overline{Y}) + [E(\overline{Y})]^2$.

Hence, $E(Y_i^2) = \mathrm{Var}(Y_i) + [E(Y_i)]^2 = \lambda + \lambda^2$. As for $E(\overline{Y}^2)$, since $Y_1, \ldots, Y_n$ are independent and identically distributed, we find

$$\mathrm{Var}(\overline{Y}) = \frac{1}{n^2}\left(\mathrm{Var}(Y_1) + \cdots + \mathrm{Var}(Y_n)\right) = \frac{n\,\mathrm{Var}(Y_1)}{n^2} = \frac{\mathrm{Var}(Y_1)}{n} = \frac{\lambda}{n}$$

so that $E(\overline{Y}^2) = \lambda/n + \lambda^2$. Combining everything gives

$$\begin{aligned} E(S^2) &= \frac{1}{n-1}\left(\sum_{i=1}^{n} E(Y_i^2) - nE(\overline{Y}^2)\right) = \frac{1}{n-1}\left(\sum_{i=1}^{n}(\lambda+\lambda^2) - n(\lambda/n+\lambda^2)\right) \\ &= \frac{1}{n-1}\left(n(\lambda+\lambda^2) - n(\lambda/n+\lambda^2)\right) = \frac{1}{n-1}(n\lambda - \lambda) = \lambda. \end{aligned}$$

which shows that S^2 is an unbiased estimator of λ.

4. (a) (i) Since

$$\begin{aligned} E(\hat{\theta}) &= E\left(\frac{\alpha}{2}Y_1 + (\alpha-1)Y_2 + \frac{\alpha}{2}Y_3\right) = \frac{\alpha}{2}E(Y_1) + (\alpha-1)E(Y_2) + \frac{\alpha}{2}E(Y_3) \\ &= \frac{\alpha}{2}4\theta + (\alpha-1)(-\theta) + \frac{\alpha}{2}(-2\theta) = 2\alpha\theta - \alpha\theta + \theta - \alpha\theta = \theta \end{aligned}$$

we conclude that the bias of $\hat{\theta}$ is 0.

4. (a) (ii) Since Y_1, Y_2, and Y_3, are independent, we find

$$\begin{aligned} \mathrm{Var}(\hat{\theta}) &= \mathrm{Var}\left(\frac{\alpha}{2}Y_1 + (\alpha-1)Y_2 + \frac{\alpha}{2}Y_3\right) = \frac{\alpha^2}{2^2}\mathrm{Var}(Y_1) + (\alpha-1)^2\,\mathrm{Var}(Y_2) + \frac{\alpha^2}{2^2}\mathrm{Var}(Y_3) \\ &= \frac{\alpha^2}{2^2}(1) + (\alpha-1)^2(4) + \frac{\alpha^2}{2^2}(3) = \alpha^2 + 4(\alpha-1)^2 = 5\alpha^2 - 8\alpha + 4. \end{aligned}$$

4. (b) In order to find the value of α that minimizes $\mathrm{Var}(\hat{\theta})$, we must minimize the function $h(\alpha) = 5\alpha^2 - 8\alpha + 4$. This is easy with the second derivative test. Since $h'(\alpha) = 10\alpha - 8$, and $h''(\alpha) = 10 > 0$, we conclude the minimal α is $\alpha = 4/5$.

5. (a) Given random variables $Y_1, \ldots, Y_n$, a statistic is simply a single-valued function $g(Y_1, \ldots, Y_n)$ of those random variables. The dual nature of the term arises as follows. Consider an experiment. Before the experiment is performed, the outcome is unknown, and after the experiment is performed, the outcome is, of course, known. Therefore, $g(Y_1, \ldots, Y_n)$ is unknown in advance and is *a priori* a random variable. Once the experiment is performed and the data $y_1, \ldots, y_n$ are known, $g(y_1, \ldots, y_n)$ is simply a number summarizing that data.

5. (b) Consider a population. A parameter is a number which summarizes the population. In general, this number is unknown. Suppose that a random sample $Y_1, \ldots, Y_n$ is drawn. An estimator is any single-valued function $g(Y_1, \ldots, Y_n)$. Once the random variables are observed so that they form the data set, say $y_1, \ldots, y_n$, they can be substituted into $g(y_1, \ldots, y_n)$ to produce a single number. This number is our predictor, or estimator, of the parameter.