Stat 252.01 Winter 2006
Assignment #10 Solutions

1. (a) Recall that the generalized likelihood ratio test for the simple null hypothesis $H_0 : \theta = \theta_0$ against the composite alternative hypothesis $H_A : \theta \neq \theta_0$ has rejection region $\{\Lambda \le c\}$ where Λ is the generalized likelihood ratio

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta}_{\text{MLE}})}$$

where $L(\theta)$ is the likelihood function. In this instance,

$$L(\theta) = \theta^n \exp\left(-\theta \sum_{i=1}^{n} y_i\right)$$

so that

$$\Lambda = \frac{\theta_0^n \exp\left(-\theta_0 \sum_{i=1}^{n} y_i\right)}{\hat{\theta}_{\text{MLE}}^n \exp\left(-\hat{\theta}_{\text{MLE}} \sum_{i=1}^{n} y_i\right)} = \left(\frac{\theta_0}{1/\overline{Y}}\right)^n \exp\left(-\theta_0 \sum y_i + 1/\overline{Y} \cdot \sum y_i\right)$$
$$= \left(\theta_0 \overline{Y}\right)^n \exp\left(n - n\theta_0 \overline{Y}\right) = e^n\, \theta_0^n\, \overline{Y}^{\,n} \exp\left(-n\theta_0 \overline{Y}\right) = e^n\, \theta_0^n \left[\overline{Y} \exp\left(-\theta_0 \overline{Y}\right)\right]^n$$

Hence, we see that the rejection region $\{\Lambda \le c\}$ can be expressed as

$$\left\{e^n\, \theta_0^n \left[\overline{Y} \exp\left(-\theta_0 \overline{Y}\right)\right]^n \le c\right\} = \left\{\overline{Y} \exp\left(-\theta_0 \overline{Y}\right) \le c^{1/n} e^{-1}\, \theta_0^{-1}\right\}$$
$$= \left\{\overline{Y} \exp\left(-\theta_0 \overline{Y}\right) \le C\right\}$$

(To be explicit, the *suitable constant* is $C = c^{1/n} e^{-1}\, \theta_0^{-1}$, although this was "not required.")

1. (b) We saw in class that $-2 \log \Lambda \sim \chi_1^2$ (approximately). This means that the generalized likelihood ratio test rejection region is $\{\Lambda \le c\} = \{-2 \log \Lambda \ge K\}$ where K is (yet another) constant. As we found above,

$$\Lambda = e^n\, \theta_0^n \left[\overline{Y} \exp\left(-\theta_0 \overline{Y}\right)\right]^n$$

so that

$$-2 \log \Lambda = -2n - 2n \log \theta_0 - 2n \log \overline{Y} + 2n\theta_0 \overline{Y}.$$

Hence, to conduct the GLRT, we need to compare the observed value of $-2 \log \Lambda$ with the appropriate chi-squared critical value which is $\chi^2_{1,0.10} = 2.70554$. Since

$$-2 \cdot 10 - 2 \cdot 10 \log 1 - 2 \cdot 10 \cdot \log 1.25 + 2 \cdot 10 \cdot 1 \cdot 1.25 \approx 2.76856$$

is the observed value of $-2 \log \Lambda$, we reject H_0 at significance level 0.10. (Note, however, that since $\chi^2_{1,0.05} = 3.84146$, we fail to reject H_0 at significance level 0.05. Again, this is for your edification, and was "not required.")

2. (a) Recall that the generalized likelihood ratio test for the simple null hypothesis $H_0 : \theta = \theta_0$ against the composite alternative hypothesis $H_A : \theta \neq \theta_0$ has rejection region $\{\Lambda \leq c\}$ where Λ is the generalized likelihood ratio

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta}_{\text{MLE}})}$$

where $L(\theta)$ is the likelihood function. In this instance,

$$L(\theta) = \theta^{2n} \left(\prod_{i=1}^{n} y_i \right) \exp\left(-\theta \sum_{i=1}^{n} y_i \right)$$

so that

$$\Lambda = \frac{\theta_0^{2n} \left(\prod_{i=1}^{n} y_i \right) \exp\left(-\theta_0 \sum_{i=1}^{n} y_i \right)}{\hat{\theta}_{\text{MLE}}^{2n} \left(\prod_{i=1}^{n} y_i \right) \exp\left(-\hat{\theta}_{\text{MLE}} \sum_{i=1}^{n} y_i \right)} = \left(\frac{1}{2/\overline{Y}} \right)^{2n} \exp\left(-\sum y_i + 2/\overline{Y} \cdot \sum y_i \right)$$

$$= \left(\frac{\overline{Y}}{2} \right)^{2n} \exp\left(2n - n\overline{Y} \right).$$

2. (b) We saw in class that $-2 \log \Lambda \sim \chi_1^2$ (approximately). This means that the generalized likelihood ratio test rejection region is $\{\Lambda \leq c\} = \{-2 \log \Lambda \geq K\}$ where $K = -2 \log c$ is (yet another) constant. (In fact, $K = \chi^2_{1,\alpha}$.) As we found above,

$$\Lambda = \left(\frac{\overline{Y}}{2} \right)^{2n} \exp\left(2n - n\overline{Y} \right)$$

so that

$$-2 \log \Lambda = -4n \log \overline{Y} + 4n \log 2 - 4n + 2n\overline{Y}.$$

Hence, to conduct the GLRT, we need to compare the observed value of $-2 \log \Lambda$ with the appropriate chi-squared critical value which is $\chi^2_{1,0.05} = 3.84146$. Since

$$-4 \cdot 5 \cdot \log 1 + 4 \cdot 5 \cdot \log 2 - 4 \cdot 5 + 2 \cdot 5 \cdot 1 \approx 3.8629$$

is the observed value of $-2 \log \Lambda$, we reject H_0 at significance level 0.05 (but just barely).