Stat 252.01 Winter 2006
Assignment #9 Solutions

1. (a) If $\beta = P(\text{Type II error})$, then we define the power of a test to be

$$\text{power} = 1 - \beta = 1 - P_{H_A}(\text{accept } H_0) = P_{H_A}(\text{reject } H_0).$$

Now, with the alternative given, we reject H_0 if

$$Z = \frac{\overline{Y} - \mu_0}{\sigma/\sqrt{n}} > 1.65,$$

or, equivalently, if

$$\overline{Y} > \sigma/\sqrt{n} \cdot 1.65 + \mu_0$$

where $\overline{Y} \sim \mathcal{N}(\mu, \sigma^2/n)$. Hence, we conclude that

$$\text{power} = P_{H_A}(\overline{Y} > \sigma/\sqrt{n} \cdot 1.65 + \mu_0).$$

Finally, if $\mu_0 = 0$, $\sigma^2 = 25$, $n = 4$, then

- $\mu = 1$:

$$\text{power} = P_{\mu=1}\left(\overline{Y} > 4.125\right) = P\left(\frac{\overline{Y} - 1}{5/2} > \frac{4.125 - 1}{5/2}\right) = 0.1056;$$

- $\mu = 2$:

$$\text{power} = P_{\mu=2}\left(\overline{Y} > 4.125\right) = P\left(\frac{\overline{Y} - 2}{5/2} > \frac{4.125 - 2}{5/2}\right) = 0.1977;$$

- $\mu = 3$:

$$\text{power} = P_{\mu=3}\left(\overline{Y} > 4.125\right) = P\left(\frac{\overline{Y} - 3}{5/2} > \frac{4.125 - 3}{5/2}\right) = 0.3264;$$

where all calculations were made using Table 4. Notice that as μ gets farther away from 0, the probability of a *correct* decision increases; it becomes easier to distinguish between the null and alternative.

(b) By mimicking the computations above, if

$$\text{power} = P_{H_A}(\overline{Y} > \sigma/\sqrt{n} \cdot 1.65 + \mu_0),$$

where $\mu_0 = 0$, $\sigma^2 = 25$, $n = 16$, then

- $\mu = 1$:

$$\text{power} = P\left(\frac{\overline{Y} - 1}{5/4} > \frac{2.0625 - 1}{5/4}\right) = 0.1977;$$

- $\mu = 2$:

$$\text{power} = P\left(\frac{\overline{Y} - 2}{5/4} > \frac{2.0625 - 2}{5/4}\right) = 0.4801;$$

- $\mu = 3$:

$$\text{power} = P\left(\frac{\overline{Y} - 3}{5/4} > \frac{2.0625 - 3}{5/4}\right) = 0.7734.$$

The reason that the power is significantly higher in **(b)** arises from the fact that as the sample size increases, the variance of $\overline{Y}$ *decreases*. (That is, larger n yields less variance.) Thus, again, as μ gets farther away from 0, the probability of a *correct* decision increases; it becomes easier to distinguish between the null and alternative. *For a fixed* H_A, by increasing n (so that there is more data), it becomes easier to detect that H_0 is false.

2. (a) If this test is to have significance level 0.1, then

$$P_{H_0}(\text{reject } H_0) = P_{\sigma=1}(S^2 > c) = 0.1.$$

Since $\sigma = 1$ under H_0 so that $9S^2$ has a χ^2 distribution with $df = 9$, we must find c so that

$$P\left(9S^2 > 9c\right) = 0.1.$$

Using the table we find that for $\alpha = 0.10$, the appropriate critical value is $\chi^2_\alpha = 14.68$. Thus, we find $9c = 14.68$ so that $c = 14.68/9 \approx 1.63$.

(b) Refer to problem **#1** where it is shown in detail that

$$\text{power} = P_{H_A}(\text{reject } H_0).$$

As in part **(a)**, we find

$$\text{power} = P_{H_A}(S^2 > 1.63) = P_{H_A}\left(\frac{9S^2}{\sigma^2} > \frac{14.68}{\sigma^2}\right),$$

where

$$\frac{9S^2}{\sigma^2} \sim \chi^2_9$$

for any σ^2. Thus,

- $\sigma^2 = 2$:
$$\text{power} = P_{\sigma^2=2}(S^2 > 1.63) = P\left(\frac{9S^2}{2} > \frac{14.68}{2}\right) \approx 0.60;$$
- $\sigma^2 = 3$:
$$\text{power} = P_{\sigma^2=3}(S^2 > 1.63) = P\left(\frac{9S^2}{3} > \frac{14.68}{3}\right) \approx 0.85;$$

where all calculations were made to the accuracy permitted by the table attached to the assignment.

3. As always, $\alpha = P_{H_0}(\text{reject } H_0)$ and $\beta = P_{H_A}(\text{accept } H_0)$. Since X has an Exponential(λ) distribution so that $f(x|\lambda) = \lambda^{-1}e^{-x/\lambda}$, and since our rejection region is $\{X < c\}$, we find that

$$\alpha = P_{H_0}(\text{reject } H_0) = P(X < c|\lambda = 1) = \int_0^c e^{-x}\,dx = 1 - e^{-c}$$

and

$$\beta = P_{H_A}(\text{accept } H_0) = P(X > c|\lambda = 1/2) = \int_c^\infty 2e^{-2x}\,dx = e^{-2c}.$$

Thus, $\alpha = 1 - e^{-c}$ and $\beta = e^{-2c}$ which easily implies that

$$1 - \alpha = \sqrt{\beta}.$$

Rewrite this as $\alpha + \sqrt{\beta} = 1$ to illustrate the direct tradeoff between them: as α increases, β must decrease, and vice-versa.

4. If X has an Exponential(λ) distribution so that $f(x|\lambda) = \lambda^{-1}e^{-x/\lambda}$, then the Fisher information (as done in class on January 26) is

$$I(\lambda) = \frac{1}{\lambda^2}$$

and the maximum likelihood estimator (as done in class on January 31) is

$$\hat{\lambda}_{\text{MLE}} = \overline{X}.$$

Hence, a significance level 0.1 test of $H_0 : \lambda = 1/5$ vs. $H_A : \lambda \neq 1/5$ has rejection region

$$\sqrt{nI(\hat{\lambda}_{\text{MLE}})}\,\left|\hat{\lambda}_{\text{MLE}} - \lambda_0\right| \geq z_{0.05}$$

or

$$\frac{\sqrt{n}}{\overline{X}}\left|\overline{X} - \frac{1}{5}\right| \geq 1.645.$$

(10.10) Let μ denote the average hardness index. In order to test the manufacturer's claim, we want to test $H_0 : \mu \geq 64$ against $H_A : \mu < 64$. It is equivalent to test $H_0 : \mu = 64$ against $H_A : \mu < 64$. The test statistic is given by

$$Z = \frac{\overline{Y} - \mu_0}{\sigma/\sqrt{n}} = \frac{62 - 64}{8/\sqrt{50}} \approx -1.77.$$

In order to conduct this test at the significance level $\alpha = 0.01$, we find that the rejection region is

$$RR = \{\text{reject } H_0 \text{ if } Z < z_{0.01} = -2.326\}.$$

Since $Z = -1.77$ does not fall in the rejection region $(-1.77 > -2.326)$, we do not reject H_0 in favour of H_A at the 0.01 level. Thus, we conclude that there is insufficient evidence to reject the manufacturer's claim.

(10.38) The rejection region is

$$\frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} < -z_\alpha$$

which is true if and only if

$$\hat{\theta} + z_\alpha \sigma_{\hat{\theta}} < \theta_0.$$

That is, H_0 will be rejected at the significance level α if and only if the $100(1-\alpha)\%$ upper confidence bound for θ (namely, $\hat{\theta} + z_\alpha \sigma_{\hat{\theta}}$) is less than θ_0.

(10.50) A t-test can be used whenever one wants to conduct a hypothesis test of the population mean when the population is known to have a normal distribution with unknown variance. The t-test also works reasonably well for populations whose distribution is mound-shaped (and resembles the normal).

(10.73) Let σ denote the standard deviation of the accuracy of the precision instrument. In order to assess the precision, we want to test $H_0 : \sigma = 0.7$ against $H_A : \sigma > 0.7$. It is equivalent to test $H_0 : \sigma^2 = 0.49$ against $H_A : \sigma^2 > 0.49$. The sample variance is given by

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \overline{y})^2 = \frac{(353-352.5)^2 + (351-352.5)^2 + (351-352.5)^2 + (355-352.5)^2}{3}$$
$$\approx 3.667$$

so that the test statistic is

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} \approx \frac{3 \cdot 3.667}{0.49} \approx 22.45.$$

(Recall that this is the distribution of the null hypothesis, i.e., assuming that $\sigma_0^2 = 0.49$.) Consulting Table 6 for the χ^2 density, we find $\chi^2_{0.005,3} = 12.8381$. Since $22.45 \gg 12.8381$, we see that the p-value must be smaller than 0.005.