Stat 252.01 Winter 2006
Assignment #4 Solutions

(8.10) (a) If $\hat{\theta} = \max(Y_1, \ldots, Y_n)$, then its distribution function is

$$F(t) = \theta^{-n\alpha}\, t^{n\alpha}, \quad 0 \le t \le \theta.$$

so that

$$f(t) = n\,\alpha\,\theta^{-n\alpha}\, t^{n\alpha-1}, \quad 0 \le t \le \theta$$

We easily calculate that

$$\mathbb{E}(\hat{\theta}) = \int_0^\theta n\,\alpha\,\theta^{-n\alpha}\, t^{n\alpha}\, dt = \frac{n\,\alpha\,\theta^{-n\alpha}\,\theta^{n\alpha+1}}{n\alpha+1} = \frac{n\,\alpha}{n\,\alpha+1}\,\theta.$$

Thus, we conclude $\hat{\theta}$ is a biased estimator of θ.

(b) Clearly, the estimator

$$\frac{n\,\alpha+1}{n\,\alpha}\,\hat{\theta} = \frac{n\,\alpha+1}{n\,\alpha}\,\max(Y_1, \ldots, Y_n)$$

is an unbiased estimator of θ.

(c) In order to calculate $\text{MSE}(\hat{\theta})$ we must find $\text{Var}(\hat{\theta})$. We find

$$\mathbb{E}(\hat{\theta}^2) = \int_0^\theta n\,\alpha\,\theta^{-n\alpha}\, t^{n\alpha+1}\, dt = \frac{n\,\alpha\,\theta^{-n\alpha}\,\theta^{n\alpha+2}}{n\alpha+2} = \frac{n\,\alpha}{n\,\alpha+2}\,\theta^2.$$

Thus,

$$\text{Var}(\hat{\theta}) = \mathbb{E}(\hat{\theta}^2) - [\mathbb{E}(\hat{\theta})]^2 = \frac{n\,\alpha}{n\,\alpha+2}\,\theta^2 - \left[\frac{n\,\alpha}{n\,\alpha+1}\,\theta\right]^2 = \frac{n\alpha}{(n\alpha+1)^2(n\alpha+2)}\,\theta^2.$$

Finally,

$$\text{MSE}(\hat{\theta}) = \text{Var}(\hat{\theta}) + [B(\hat{\theta})]^2 = \left[\frac{n\alpha}{(n\alpha+1)^2(n\alpha+2)} + \frac{1}{(n\alpha+1)^2}\right]\theta^2 = \frac{2\theta^2}{(n\alpha+1)(n\alpha+2)}.$$

(8.15) If $Y_{(1)} = \min(Y_1, \ldots, Y_n)$, then its distribution function is

$$F(t) = 1 - e^{-tn/\theta}, \quad t > 0$$

so that

$$f(t) = \frac{n}{\theta} e^{-tn/\theta}, \quad t > 0.$$

We easily calculate (use integration by parts) that

$$\mathbb{E}(Y_{(1)}) = \int_0^\infty \frac{n}{\theta} t e^{-tn/\theta} dt = \frac{\theta}{n}$$

so that if $\hat{\theta} = nY_{(1)}$, then $\mathbb{E}(\hat{\theta}) = n\mathbb{E}(Y_{(1)}) = \theta$ so that $\hat{\theta}$ is an unbiased estimator of θ.

In order to calculate $\text{MSE}(\hat{\theta})$ we must find $\text{Var}(\hat{\theta})$. Notice, however, that $Y_{(1)}$ is an exponential random variable with parameter θ/n. Thus,

$$\text{Var}(\hat{\theta}) = \text{Var}(nY_{(1)}) = n^2\,\text{Var}(Y_{(1)}) = n^2\frac{\theta^2}{n^2} = \theta^2.$$

This gives

$$\text{MSE}(\hat{\theta}) = \text{Var}(\hat{\theta}) + [B(\hat{\theta})]^2 = \theta^2 + 0 = \theta^2.$$

(8.30) If $\hat{\lambda} = \overline{Y}$, then $\mathbb{E}(\hat{\lambda}) = \mathbb{E}(\overline{Y}) = \lambda$ so that $\overline{Y}$ is an unbiased estimator of λ. Since the standard error of $\hat{\lambda}$ is

$$\sigma_{\hat{\lambda}} = \sqrt{V(\overline{Y})} = \sqrt{\frac{\lambda}{n}},$$

a natural guess for the estimated standard error is

$$\hat{\sigma}_{\hat{\lambda}} = \sqrt{\frac{\hat{\lambda}}{n}}.$$

(8.32) If $\hat{\theta} = \overline{Y}$, then $\mathbb{E}(\hat{\theta}) = \mathbb{E}(\overline{Y}) = \theta$ so that $\overline{Y}$ is an unbiased estimator of θ. The standard error of $\hat{\theta}$ is

$$\sigma_{\hat{\theta}} = \sqrt{V(\overline{Y})} = \frac{\theta}{\sqrt{n}}.$$

Thus, if the estimated standard error is

$$\hat{\sigma}_{\hat{\theta}} = \frac{\hat{\theta}}{\sqrt{n}},$$

then

$$\mathbb{E}(\hat{\sigma}_{\hat{\theta}}) = \frac{\mathbb{E}(\hat{\theta})}{\sqrt{n}} = \frac{\theta}{\sqrt{n}} = \sigma_{\hat{\theta}}$$

so that $\hat{\sigma}_{\hat{\theta}}$ is an unbiased estimator of the standard error.