**Statistics 252 Winter 2005 Midterm #1 – Solutions**

**1. (a)** Since

$$\log f(y|\theta) = \log(y) - 2\log(\theta) - \frac{y^2}{2\theta^2},$$

we find

$$\frac{\partial^2}{\partial\theta^2}\log f(y|\theta) = \frac{2}{\theta^2} - \frac{3y^2}{\theta^4}.$$

Thus,

$$I(\theta) = -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\log f(Y|\theta)\right) = \frac{3\mathbb{E}(Y^2)}{\theta^4} - \frac{2}{\theta^2} = \frac{4}{\theta^2}.$$

**(b)** To find $\hat{\theta}_{\text{MOM}}$ we solve the equation $\mathbb{E}(Y) = \overline{Y}$ for $\theta$. This implies

$$\hat{\theta}_{\text{MOM}} = \sqrt{\frac{2}{\pi}}\,\overline{Y}.$$

**(c)**

$$\begin{aligned}\operatorname{Var}(\hat{\theta}_{\text{MOM}}) = \frac{2}{\pi}\operatorname{Var}\overline{Y} = \frac{2}{n\pi}\operatorname{Var}Y_1 = \frac{2}{n\pi}\left(\mathbb{E}(Y_1^2) - [\mathbb{E}(Y)]^2\right) &= \frac{2}{n\pi}\left(2 - \frac{\pi}{2}\right)\theta^2 \\ &= \left(\frac{4-\pi}{n\pi}\right)\theta^2\end{aligned}$$

**(d)** The likelihood function is

$$L(\theta) = \prod_{i=1}^n f(y_i|\theta) = \left(\prod_{i=1}^n y_i\right)\theta^{-2n}\exp\left(-\frac{1}{2\theta^2}\sum_{i=1}^n y_i^2\right)$$

so that the log-likelihood function is

$$\ell(\theta) = \sum_{i=1}^n \log(y_i) - 2n\log(\theta) - \frac{1}{2\theta^2}\sum_{i=1}^n y_i^2.$$

Hence $\ell'(\theta) = 0$ implies

$$0 = -\frac{2n}{\theta} + \frac{1}{\theta^3}\sum_{i=1}^n y_i^2$$

so that

$$\hat{\theta}_{\text{MLE}} = \sqrt{\frac{1}{2n}\sum_{i=1}^n Y_i^2}.$$

**(e)** An approximate 95% confidence interval for $\theta$ is given by

$$\hat{\theta}_{\text{MLE}} \pm 1.96\,\frac{1}{\sqrt{n\,I(\hat{\theta}_{\text{MLE}})}}.$$

Since $n = 100$ and $\sum y_i^2 = 80000$, we conclude that

$$\hat{\theta}_{\text{MLE}} = \sqrt{\frac{80000}{200}} = \sqrt{400} = 20$$

and

$$I(\hat{\theta}_{\text{MLE}}) = \frac{4}{\hat{\theta}^2_{\text{MLE}}} = \frac{4}{400} = \frac{1}{100}.$$

Hence, an approximate 95% confidence interval for $\theta$ is

$$20 \pm 1.96.$$

**2. (a)** Consider a population described by an unknown parameter of interest. An estimator is a rule for constructing a "guess" or "estimate" of the parameter based on a random sample from the population. Hence, an estimator is a random variable. After the data have been observed, it is possible to evaluate the estimator using that data. This is then called an estimate of the parameter.

**(b)** Suppose that $\hat{\theta}$ is an estimator of $\theta$. The random interval $[L(\hat{\theta}), U(\hat{\theta})]$ is a 93% confidence interval for $\theta$ if

$$P(L(\hat{\theta}) \leq \theta \leq U(\hat{\theta})) = 0.93.$$

Hence, we interpret a 93% confidence interval to mean that before the data have been observed, there is a 93% chance that the parameter will lie in the random interval. However, once the data have been observed, no such probability statement is true. Either the given interval does or does not contain $\theta$. Alternatively, if many, many intervals are observed, each constructed using the same formula, then the long-run average that will contain $\theta$ is 0.93.

**3. (a)** Since

$$\mathbb{E}(\hat{\theta}_1) = \frac{1}{4}\mathbb{E}(X) + \frac{1}{2}\mathbb{E}(Y) = \frac{2\theta}{4} + \frac{\theta}{2} = \theta$$

and

$$\mathbb{E}(\hat{\theta}_2) = \mathbb{E}(X) - \mathbb{E}(Y) = 2\theta - \theta = \theta$$

we conclude that $B(\hat{\theta}_1) = B(\hat{\theta}_2) = 0$. Thus,

$$\text{MSE}(\hat{\theta}_1) = \text{Var}(\hat{\theta}_1) = \frac{1}{16}\,\text{Var}(X) + \frac{1}{4}\,\text{Var}(Y) = \frac{3}{4}$$

and

$$\text{MSE}(\hat{\theta}_2) = \text{Var}(\hat{\theta}_2) = \text{Var}(X) + \text{Var}(Y) = 6.$$

**(b)** We find

$$\text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\text{Var}(\hat{\theta}_2)}{\text{Var}(\hat{\theta}_1)} = 8.$$

Since both $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased, the one with the smaller variance is preferrable, namely $\hat{\theta}_1$.

**(c)** Since

$$\mathbb{E}(\hat{\theta}_c) = \frac{c}{2}\mathbb{E}(X) + (1-c)\mathbb{E}(Y) = \frac{2c\theta}{2} + (1-c)\theta = \theta$$

we see $\hat{\theta}_c$ is unbiased. Since

$$\text{Var}(\hat{\theta}_c) = \frac{c^2}{4}\,\text{Var}(X) + (1-c)^2\,\text{Var}(Y) = c^2 + 2(1-c)^2 = 3c^2 - 4c + 2$$

the value that minimizes $\text{Var}(\hat{\theta}_c)$ is the same value that minimizes the polynomial $g(c) = 3c^2 - 4c + 2$. Since $g'(c) = 0$ implies $c = 2/3$, and since $g''(2/3) > 0$, the minimal value of $c$ is $2/3$.

**4. (a)** We find that

$$\mathbb{E}(\overline{Y}) = \mathbb{E}(Y_1) = 252\,\theta.$$

Thus, if

$$\hat{\theta}_A = \frac{\overline{Y}}{252} = \frac{1}{252n}\sum_{i=1}^{n} Y_i$$

then $\hat{\theta}_A$ is an unbiased estimator of $\theta$.

**(b)** Since

$$\log f(y|\theta) = -252\log(\theta) - \log(251!) + 251\log(y) - \frac{y}{\theta},$$

we find

$$\frac{\partial^2}{\partial\theta^2}\log f(y|\theta) = \frac{252}{\theta^2} - \frac{2y}{\theta^3}.$$

Thus,

$$I(\theta) = -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\log f(Y|\theta)\right) = -\frac{252}{\theta^2} + \frac{2\mathbb{E}(Y)}{\theta^3} = \frac{252}{\theta^2}.$$

**(c)** The Cramer-Rao inequality tells us that an unbiased estimator $\hat{\theta}$ of $\theta$ must satisfy

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{nI(\theta)} = \frac{\theta^2}{252n}.$$

Since

$$\mathrm{Var}(\hat{\theta}_A) = \frac{1}{252^2 n}\,\mathrm{Var}\,Y_1 = \frac{1}{252^2}\cdot(252\,\theta^2) = \frac{\theta^2}{252n},$$

we have found an unbiased estimator whose variance attains the lower bound of the Cramer-Rao inequality. Hence, $\hat{\theta}_A$ must be the MVUE of $\theta$.