Statistics 252 “Practice” Midterm – Winter 2005

This exam has 5 problems and is worth 50 points.

You have 50 minutes to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements.*

Calculators are permitted, as well as an $8\frac{1}{2} \times 11$ *double-sided page of handwritten notes. A dictionary will be provided.*

Name: ____________________________________

Instructor: Michael Kozdron

Problem	Score
1	
2	
3	
4	
5	

TOTAL: ____________

Note that all of these problems appeared on previous exams, and are relevant for this semester’s course. However, completing this “practice” exam should constitute only a portion of your study and preparation for the midterm.

1. (*8 points*) Consider a random variable Y with density function

$$f(y|\theta) = \theta^2 e^{-\theta^2 y},$$

where $x > 0$, and $\theta > 0$.

(a) Compute the Fisher information $I(\theta)$ for this density.

Assume that $0 < \alpha_1 < \frac{1}{2}$ and $0 < \alpha_2 < \frac{1}{2}$.

(b) Using the pivotal quantity $\theta^2 y$, verify that

$$\left(\frac{-\log(1-\alpha_1)}{Y}, \frac{-\log(\alpha_2)}{Y} \right)$$

is a confidence interval for θ with coverage probability $1 - (\alpha_1 + \alpha_2)$.

2. (*10 points*) Assume that $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed random variables from the Pareto distribution, which has density function

$$f(y|\theta) = (\theta - 1)y^{-\theta}$$

for $0 \le y < \infty$, and $2 \le \theta < \infty$.

(a) Verify that the maximum likelihood estimator of θ is

$$\hat{\theta}_{\text{MLE}} = 1 + \frac{n}{\sum_{i=1}^{n} \log Y_i}.$$

(b) Verify that the Fisher Information in a single observation is

$$I(\theta) = \frac{1}{(\theta - 1)^2}.$$

(c) Say $n = 25$, and we observe a sample for which

$$\sum_{i=1}^{25} \log y_i = 5.$$

Give an approximate 90% confidence interval for θ. (The 0.8, 0.9. 0.95 quantiles for the normal distribution are 0.84, 1.28, 1.64, respectively.)

3. (*8 points*) A random variable Y has the Laplace distribution if its density function

$$f(y|\lambda,\theta) = \frac{\lambda}{2}\, e^{-\lambda|y-\theta|}, \quad 0 < y < \infty$$

with parameters $\lambda > 0$ and $-\infty < \theta < \infty$. The expectation and variance of Y are

$$\mathbb{E}(Y) = \theta \quad \text{and} \quad \text{Var}(Y) = \frac{2}{\lambda^2}.$$

Find the method of moments estimators of both λ and θ.

(For extra practice, try and derive $\mathbb{E}(Y)$ *and* $\text{Var}(Y)$ *directly.)*

4. (*8 points*)

(a) In the context of Stat 252, clearly define what is meant by *parameter* and *estimator.*

(b) Why do you think it is desirable to find a *minimum variance unbiased estimator*?

5. (*16 points*) Suppose that the random variables $Y_1, Y_2, \ldots, Y_{10}$ are independent and identically distributed Uniform$(0, \theta)$ random variables. That is, each Y_i has density

$$f(y|\theta) = \begin{cases} \frac{1}{\theta}, & 0 \le y \le \theta, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $\hat{\theta}_{\text{MOM}}$, the method of moments estimator of θ. Show that $\hat{\theta}_{\text{MOM}}$ is an unbiased estimator of θ.

(b) From previous work, we know that $\hat{\theta}_{\text{MLE}}$, the maximum likelihood estimator of θ, is

$$\hat{\theta}_{\text{MLE}} = \max(Y_1, \ldots, Y_{10}).$$

Find an unbiased estimator which is a function of the MLE. Call it $\hat{\theta}_B$.

(c) Find the efficiency of $\hat{\theta}_{\text{MOM}}$ relative to $\hat{\theta}_B$. Which estimator do you prefer? Why?

(d) Find the Fisher Information in a single observation from the Uniform$(0, \theta)$ distribution.

(e) Based on your previous answers, what can you say about the minimum variance unbiased estimator (MVUE) of θ? Why is the Cramer-Rao inequality of no use in this case?