Statistics 252 Midterm #1 – Winter 2005

This exam has 4 problems and is worth 50 points. **Instructor:** Michael Kozdron

You have 50 minutes to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements. Calculators are permitted, as well as an* $8\frac{1}{2} \times 11$ *double-sided page of handwritten notes.*

1. (*18 points*) A continuous random variable Y is said to have the Rayleigh(θ) distribution if the probability density function of Y is

$$f(y|\theta) = \frac{y}{\theta^2} \exp\left(-\frac{y^2}{2\theta^2}\right).$$

where $y > 0$ and $\theta > 0$. It turns out that

$$\mathbb{E}(Y) = \sqrt{\frac{\pi}{2}}\,\theta$$

and

$$\mathbb{E}(Y^2) = 2\theta^2.$$

(a) Determine the Fisher information $I(\theta)$ for the Rayleigh(θ) distribution.

Suppose that $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed Rayleigh(θ) random variables.

(b) Compute $\hat{\theta}_{\text{MOM}}$, the method of moments estimator of θ.

(c) Compute the variance of $\hat{\theta}_{\text{MOM}}$.

(d) Determine the maximum likelihood estimator of θ. You do *not* need to verify that your critical point is actually a maximum.

(e) Suppose that a random sample of size $n = 100$ is observed. From this random sample,

$$\sum_{i=1}^{100} y_i^2 = 80000$$

is calculated. Construct an approximate 95% confidence interval for θ.
(*Note that* $z_{0.025} = 1.96$ *is the critical value for a normal distribution-based 95% CI.*)

2. (*8 points*)

(a) Briefly describe the difference between an estimator and an estimate.

(b) Describe how to interpret a 93% confidence interval.

3. (*12 points*) Let X and Y be uncorrelated random variables. Suppose that X has mean 2θ and variance 4. Suppose that Y has mean θ and variance 2. The parameter θ is unknown.

(a) Compute the bias and mean squared error for both of the following estimators of θ:

$$\hat{\theta}_1 = \frac{1}{4}X + \frac{1}{2}Y$$
$$\hat{\theta}_2 = X - Y$$

(b) Compute the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$. Which estimator do you prefer? Why?

(c) Verify that the estimator

$$\hat{\theta}_c = \frac{c}{2}X + (1-c)Y$$

is unbiased. Find the value of c which minimizes $\mathrm{Var}(\hat{\theta}_c)$.

4. (*12 points*) Suppose that $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed random variables, each having density function

$$f(y|\theta) = \frac{\theta^{-252}}{251!}\, y^{251}\, e^{-y/\theta}$$

if $y > 0$, where $\theta > 0$ is a parameter. It is known that if $Y \sim f(y|\theta)$, then

$$\mathbb{E}(Y) = 252\,\theta \quad \text{and} \quad \mathrm{Var}(Y) = 252\,\theta^2.$$

Let $\hat{\theta} = \overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$.

(a) Find a function of $\hat{\theta}$ which is an unbiased estimator of θ. Call it $\hat{\theta}_A$

(b) Compute the Fisher information in a single observation from this density.

(c) Carefully explain why $\hat{\theta}_A$ must be the minimum variance unbiased estimator (MVUE) of θ.