Make sure that this examination has 11 numbered pages

University of Regina
Department of Mathematics & Statistics
Final Examination
200510
(April 22, 2005)

Statistics 252-001
Mathematical Statistics

Name: ______________________ **Student Number:** ______________

Instructor: Michael Kozdron **Time:** 3 hours

Read all of the following information before starting the exam.

*You have **3** hours to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. **Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.** Several problems require written explanations in context. Only complete solutions written in the context specified by the problem will be awarded full points, and points will be deducted for incoherent, incorrect, and/or irrelevant statements.*

You may use standard notation; however, any new notations or abbreviations that you introduce must be clearly defined.

*Calculators are permitted; however, you must still show all your work. You are also permitted to have **TWO** 8.5×11 yellow pages of handwritten notes (double-sided) for your personal use. Other than these exceptions, no other aids are allowed.*

Note that blank space is not an indication of a question's difficulty. The order of the test questions is essentially random; they are not intentionally written easiest-to-hardest.

*This test has **11** numbered pages with **7** questions totalling **150** points. The number of points per question is indicated.*

DO NOT WRITE BELOW THIS LINE

Problem 1	______	Problem 2	______	Problem 3	______
Problem 4	______	Problem 5	______	Problem 6	______
Problem 7	______				
				TOTAL	______

1. (*40 points*) Suppose that $Y_1, \ldots, Y_n$ are independent and identically distributed random variables with each Y_i having density function

$$f(y|\theta) = \frac{\theta^2}{y^3} \exp\left(-\theta/y\right),$$

where $y > 0$ and $\theta > 0$. It is known that $\mathbb{E}(Y_i) = \theta$ and $\mathbb{E}\left(\frac{1}{Y_i}\right) = \frac{2}{\theta}$ for each $i = 1, \ldots, n$.

(a) Determine $\hat{\theta}_{\text{MOM}}$, the method of moments estimator of θ.

(b) Compute the likelihood function $L(\theta)$ for this random sample.

(c) Show that the maximum likelihood estimator of θ is $\hat{\theta}_{\text{MLE}} = \dfrac{2n}{\sum_{i=1}^{n} \frac{1}{Y_i}}$.

(continued)

(d) Show that $\sum_{i=1}^{n} \frac{1}{Y_i}$ is a sufficient statistic for the estimation of θ.

(e) Explain why $\hat{\theta}_{\mathrm{MLE}}$ must also be a sufficient statistic for the estimation of θ.

(f) Find the Fisher information $I(\theta)$ in a single observation from this density.

(continued)

(g) Using the standard approximation for the distribution of a maximum likelihood estimator based on the Fisher information, construct an approximate 90% confidence interval for θ.

(h) Verify that the generalized likelihood ratio test for the test of the hypothesis $H_0 : \theta = \theta_0$ against $H_A : \theta \neq \theta_0$ has rejection region of the form

$$\left\{ \left(\sum_{i=1}^{n} \frac{1}{Y_i} \right)^{2n} \exp\left(-\theta_0 \sum_{i=1}^{n} \frac{1}{Y_i} \right) \leq C \right\}$$

for some constant C.

(continued)

To answer **(i)** and **(j)** below, suppose that an observation of size $n = 8$ produces

$$\sum_{i=1}^{8} \frac{1}{Y_i} = 10.$$

(i) Based on your confidence interval constructed in **(g)** and on the above data, can you reject the hypothesis $H_0 : \theta = 1$ in favour of $H_A : \theta \neq 1$ at the significance level $\alpha = 0.10$?

(j) Based on your generalized likelihood ratio test constructed in **(h)** and on the above data, can you reject the hypothesis $H_0 : \theta = 1$ in favour of $H_A : \theta \neq 1$ at the significance level $\alpha = 0.10$?

2. (*14 points*) Consider a random variable Y with density function

$$f(y|\theta) = \frac{e^{(y-\theta)}}{\left[1 + e^{(y-\theta)}\right]^2},$$

where $-\infty < y < \infty$, and $-\infty < \theta < \infty$. Using the pivotal quantity $y - \theta$, verify that if $0 < \alpha_1 < 1/2$ and $0 < \alpha_2 < 1/2$, then

$$\left(\; Y - \log\left(\frac{1-\alpha_2}{\alpha_2}\right), Y - \log\left(\frac{\alpha_1}{1-\alpha_1}\right) \; \right)$$

is a confidence interval for θ with coverage probability $1 - (\alpha_1 + \alpha_2)$.

3. (*24 points*) An electrical circuit consists of three batteries X_1, X_2, X_3 connected in series to a lightbulb Y. We model the battery lifetimes as independent and identically distributed Exponential(λ) random variables. Our experiment to measure the lifetime of the lightbulb is stopped when any one of the batteries fails. Hence, the only random variable we observe is $Y = \min\{X_1, X_2, X_3\}$.

(a) Determine the distribution of the random variable Y.

(b) Compute $\hat{\lambda}_{\text{MLE}}$, the maximum likelihood estimator of λ. *Hint:* Use the result of **(a)**.

(c) Determine the mean square error of $\hat{\lambda}_{\text{MLE}}$.

(d) Use the Cramer-Rao lower bound to prove that $\hat{\lambda}_{\text{MLE}}$ is the minimum variance unbiased estimator of λ.

4. (*16 points*)

(a) Assume that the outcome of an experiment is a single random variable X. An 80% confidence interval for a parameter θ has the form $(X-1, X+2)$. From this, determine a rejection rule for testing $H_0 : \theta = 5$ against $H_A : \theta \neq 5$ at significance level 0.2.

(b) Let Y_1 and Y_2 be independent Uniform$(0, \theta)$ random variables. Consider testing $H_0 : \theta = 1$ against $H_A : \theta > 1$ by rejecting H_0 when $\max\{Y_1, Y_2\} > c$. Find c so that this test has significance level $\frac{19}{100}$. What is the power of this test (as a function of θ)?

5. (*16 points*)

(a) Suppose you have been asked to analyze a data set, and you are planning to use a particular estimator to estimate a parameter. Explain why it is important to determine (either exactly or approximately) the sampling distribution of this estimator.

(b) Suppose that you want to compare two simple hypotheses in light of a data set. You use a test with significance level $\alpha = 0.05$. The data yield a test statistic for which you accept the null hypothesis (or, if you prefer, you do not reject the null hypothesis). You would like to use this result to argue that the null hypothesis is likely to be true. What additional information about the test might you want first? Explain.

6. (*16 points*) Suppose that $Y_1, \ldots, Y_5$ are independent and identically distributed $\mathcal{N}(\mu, \sigma^2)$ random variables. As usual, let

$$\overline{Y} = \frac{1}{5}\sum_{i=1}^{5} Y_i \quad \text{and} \quad S^2 = \frac{1}{4}\sum_{i=1}^{5}(Y_i - \overline{Y})^2.$$

It is known that $\mu = 0$, but σ^2 is unknown. You want to test $H_0 : \sigma = 1$ against $H_A : \sigma > 1$ by rejecting H_0 when $S^2 > 1.945$. (*You may consider your calculations accurate to two decimal places when consulting the appropriate table(s).*)

(a) Verify that this test has significance level $\alpha = 0.10$.

(b) Using the test determined in **(a)**, find the power of the test when $\sigma = 2.7$.

7. (*24 points*) The goal of this problem is to study the *simple quadratic regression model.* If the random variable Y is defined by

$$Y = b_0 + b_1 x^2 + \varepsilon,$$

where ε is a random variable with mean 0, and x is a real (i.e., *not* random) variable, then we say that Y follows the simple quadratic regression model. The *least squares parabola* is given by

$$\hat{y} = \hat{b}_0 + \hat{b}_1 x^2.$$

If real variable inputs $x_1, \ldots, x_n$ produce observations $y_1, \ldots, y_n$, respectively, then the least squares estimators $\hat{b}_0$ and $\hat{b}_1$ are chosen to minimize the sum of the squares of the errors

$$\mathrm{SSE}(\hat{b}_0, \hat{b}_1) = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - \hat{b}_0 - \hat{b}_1 x_i^2)^2.$$

Find $\hat{b}_0$ and $\hat{b}_1$, the least squares estimators of b_0 and b_1, respectively. (You do not need to check the second derivative test.)

(*The end. Enjoy your summer!*)