Stat 252.01 Winter 2005
Assignment #7 Solutions

**1. (a)** As proved in class, the least squares estimate $\hat{\beta}_0$ is given by

$$\hat{\beta}_0 = \overline{y} - \hat{\beta}_1\overline{x}.$$

Thus,

$$\begin{aligned}\sum_{i=1}^{n}(y_i - \hat{y}_i) = \sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = \sum_{i=1}^{n} y_i - n\hat{\beta}_0 - \hat{\beta}_1\sum_{i=1}^{n} x_i &= n\overline{y} - n\hat{\beta}_0 - n\hat{\beta}_1\overline{x} \\ &= n\overline{y} - n(\overline{y} - \hat{\beta}_1\overline{x}) - n\hat{\beta}_1\overline{x} \\ &= 0.\end{aligned}$$

**(b)** To prove that the least squares line

$$\hat{y} = \hat{\beta}_0 - \hat{\beta}_1 x$$

passes throught the point $(\overline{x}, \overline{y})$ we simply evaluate the equation at the point $x = \overline{x}$ and see what happens:

$$\hat{\beta}_0 - \hat{\beta}_1\overline{x} = (\overline{y} - \hat{\beta}_1\overline{x}) - \hat{\beta}_1\overline{x} = \overline{y}.$$

That is, the point $(\overline{x}, \overline{y})$ satisfies the least squares equation.

**2. (11.1)** From the data given, we find that

$$\sum_{i=1}^{5} x_i = 0, \quad \sum_{i=1}^{5} y_i = 7.5, \quad \sum_{i=1}^{5} x_i y_i = -6, \quad \sum_{i=1}^{5} x_i^2 = 10.$$

Since

$$\hat{\beta}_1 = \frac{\displaystyle\sum_{i=1}^{5} x_i y_i - n\overline{x}\,\overline{y}}{\displaystyle\sum_{i=1}^{5} x_i^2 - n\overline{x}^2} \quad \text{and} \quad \hat{\beta}_0 = \overline{y} - \hat{\beta}_1\overline{x},$$

we conclude that

$$\hat{\beta}_1 = \frac{-6 - 5\cdot(0/5)\cdot(7.5/5)}{10 - 5\cdot(0/5)^2} = \frac{-3}{5} \quad \text{and} \quad \hat{\beta}_0 = \frac{7.5}{5} - \frac{-3}{5}\cdot\frac{0}{5} = \frac{3}{2}.$$

Hence, the equation of the least squares line is given by

$$\hat{y} = \frac{3}{2} - \frac{3}{5}x.$$

**2. (11.6)** Notice that the linear model $Y_i = \beta_1 x_i + \varepsilon_i$ is simply the usual least squares model with $\beta_0 = 0$ (called the *no-intercept model* for obvious reasons). In order to find the least squares estimate of $\beta_1$, called $\hat{\beta}_1$, we must minimze the sum of the squares of the errors,

$$\text{SSE} = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2.$$

Now, $y_i$ is the $i$th observation of the random variable $Y$, and $\hat{y}_i = \hat{\beta}_1 x_i$ (since the linear model has 0 intercept). Hence,

$$\text{SSE}(\hat{\beta}_1) = \sum_{i=1}^{n} (y_i - \hat{\beta}_1 x_i)^2.$$

To minimize $\text{SSE}(\hat{\beta}_1)$ we need to take the derivative with respect to $\hat{\beta}_1$, set it equal to zero, and solve for the critical points. That is,

$$\frac{d}{d\hat{\beta}_1}\text{SSE}(\hat{\beta}_1) = -2\sum_{i=1}^{n}(y_i - \hat{\beta}_1 x_i)x_i$$

so that

$$\frac{d}{d\hat{\beta}_1}\text{SSE}(\hat{\beta}_1) = 0$$

implies

$$0 = -2\sum_{i=1}^{n}(y_i - \hat{\beta}_1 x_i)x_i = -2\sum_{i=1}^{n} x_i y_i + 2\hat{\beta}_1 \sum_{i=1}^{n} x_i^2$$

or, in other words,

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}.$$

Since $\text{SSE}(\hat{\beta}_1)$ is a function of only one variable, the second derivative test should be checked. We find

$$\frac{d^2}{d\hat{\beta}_1^{\,2}}\text{SSE}(\hat{\beta}_1) = 2\sum_{i=1}^{n} x_i^2 > 0$$

so that the critical value $\hat{\beta}_1$ is truly a minimum.

**3.** Notice that the linear model $Y_i = \mu + \varepsilon_i$ is simply the usual least squares model with $\beta_0 = \mu$ and $\beta_1 = 0$ (often called the *random noise model*). In order to find the least squares estimate of $\mu$, called $\hat{\mu}$, we must minimze the sum of the squares of the errors,

$$\text{SSE} = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2.$$

Now, $y_i$ is the $i$th observation of the random variable $Y$, and $\hat{y}_i = \hat{\mu}$ (since the linear model has 0 slope). Hence,

$$\text{SSE}(\hat{\mu}) = \sum_{i=1}^{n}(y_i - \hat{\mu})^2.$$

To minimize $\text{SSE}(\hat{\mu})$ we need to take the derivative with respect to $\hat{\mu}$, set it equal to zero, and solve for the critical points. That is,

$$\frac{d}{d\hat{\mu}}\text{SSE}(\hat{\mu}) = -2\sum_{i=1}^{n}(y_i - \hat{\mu})$$

so that

$$\frac{d}{d\hat{\mu}}\text{SSE}(\hat{\mu}) = 0$$

implies

$$0 = -2\sum_{i=1}^{n}(y_i - \hat{\mu}) = -2\sum_{i=1}^{n} y_i + 2n\hat{\mu} = -2n\overline{y} + 2n\hat{\mu}$$

or, in other words,

$$\hat{\mu} = \overline{y}.$$

Since $\text{SSE}(\hat{\mu})$ is a function of only one variable, the second derivative test should be checked. We find

$$\frac{d^2}{d\hat{\mu}^2}\text{SSE}(\hat{\mu}) = 2n > 0$$

so that the critical value $\hat{\mu} = \overline{y}$ is truly a minimum.