Stat 252.01 Winter 2005
Assignment #5 Solutions

1. (a) If $\beta = P(\text{Type II error})$, then we define the power of a test to be

$$\text{power} = 1 - \beta = 1 - P_{H_A}(\text{accept } H_0) = P_{H_A}(\text{reject } H_0).$$

Now, with the alternative given, we reject H_0 if

$$Z = \frac{\overline{Y} - \mu_0}{\sigma/\sqrt{n}} > 1.65,$$

or, equivalently, if

$$\overline{Y} > \sigma/\sqrt{n} \cdot 1.65 + \mu_0$$

where $\overline{Y} \sim \mathcal{N}(\mu, \sigma^2/n)$. Hence, we conclude that

$$\text{power} = P_{H_A}(\overline{Y} > \sigma/\sqrt{n} \cdot 1.65 + \mu_0).$$

Finally, if $\mu_0 = 0$, $\sigma^2 = 25$, $n = 4$, then

- $\mu = 1$:
$$\text{power} = P_{\mu=1}\left(\overline{Y} > 4.125\right) = P\left(\frac{\overline{Y} - 1}{5/2} > \frac{4.125 - 1}{5/2}\right) = 0.1056;$$
- $\mu = 2$:
$$\text{power} = P_{\mu=2}\left(\overline{Y} > 4.125\right) = P\left(\frac{\overline{Y} - 2}{5/2} > \frac{4.125 - 2}{5/2}\right) = 0.1977;$$
- $\mu = 3$:
$$\text{power} = P_{\mu=3}\left(\overline{Y} > 4.125\right) = P\left(\frac{\overline{Y} - 3}{5/2} > \frac{4.125 - 3}{5/2}\right) = 0.3264;$$

where all calculations were made using Table 4. Notice that as μ gets farther away from 0, the probability of a *correct* decision increases; it becomes easier to distinguish between the null and alternative.

(b) By mimicking the computations above, if

$$\text{power} = P_{H_A}(\overline{Y} > \sigma/\sqrt{n} \cdot 1.65 + \mu_0),$$

where $\mu_0 = 0$, $\sigma^2 = 25$, $n = 16$, then

- $\mu = 1$:
$$\text{power} = P\left(\frac{\overline{Y} - 1}{5/4} > \frac{2.0625 - 1}{5/4}\right) = 0.1977;$$
- $\mu = 2$:
$$\text{power} = P\left(\frac{\overline{Y} - 2}{5/4} > \frac{2.0625 - 2}{5/4}\right) = 0.4801;$$
- $\mu = 3$:
$$\text{power} = P\left(\frac{\overline{Y} - 3}{5/4} > \frac{2.0625 - 3}{5/4}\right) = 0.7734.$$

The reason that the power is significantly higher in **(b)** arises from the fact that as the sample size increases, the variance of $\overline{Y}$ *decreases*. (That is, larger n yields less variance.) Thus, again, as μ gets farther away from 0, the probability of a *correct* decision increases; it becomes easier to distinguish between the null and alternative. *For a fixed* H_A, by increasing n (so that there is more data), it becomes easier to detect that H_0 is false.

2. **(a)** If this test is to have significance level 0.1, then

$$P_{H_0}(\text{reject } H_0) = P_{\sigma=1}(S^2 > c) = 0.1.$$

Since $\sigma = 1$ under H_0 so that $9S^2$ has a χ^2 distribution with $df = 9$, we must find c so that

$$P\left(9S^2 > 9c\right) = 0.1.$$

Using the table we find that for $\alpha = 0.10$, the appropriate critical value is $\chi^2_\alpha = 14.68$. Thus, we find $9c = 14.68$ so that $c = 14.68/9 \approx 1.63$.

(b) Refer to problem **#1** where it is shown in detail that

$$\text{power} = P_{H_A}(\text{reject } H_0).$$

As in part **(a)**, we find

$$\text{power} = P_{H_A}(S^2 > 1.63) = P_{H_A}\left(\frac{9S^2}{\sigma^2} > \frac{14.68}{\sigma^2}\right),$$

where

$$\frac{9S^2}{\sigma^2} \sim \chi^2_9$$

for any σ^2. Thus,

- $\sigma^2 = 2$:
$$\text{power} = P_{\sigma^2=2}(S^2 > 1.63) = P\left(\frac{9S^2}{2} > \frac{14.68}{2}\right) \approx 0.60;$$
- $\sigma^2 = 3$:
$$\text{power} = P_{\sigma^2=3}(S^2 > 1.63) = P\left(\frac{9S^2}{3} > \frac{14.68}{3}\right) \approx 0.85;$$

where all calculations were made to the accuracy permitted by the table attached to the assignment.

3. As always, $\alpha = P_{H_0}(\text{reject } H_0)$ and $\beta = P_{H_A}(\text{accept } H_0)$. Since X has an Exponential(λ) distribution so that $f(x|\lambda) = \lambda^{-1}e^{-x/\lambda}$, and since our rejection region is $\{X < c\}$, we find that

$$\alpha = P_{H_0}(\text{reject } H_0) = P(X < c|\lambda = 1) = \int_0^c e^{-x}\,dx = 1 - e^c$$

and

$$\beta = P_{H_A}(\text{accept } H_0) = P(X > c|\lambda = 1/2) = \int_c^\infty 2e^{-2x}\,dx = e^{-2c}.$$

Thus, $\alpha = 1 - e^c$ and $\beta = e^{-2c}$ which easily implies that

$$1 - \alpha = \sqrt{\beta}.$$

Rewrite this as $\alpha + \sqrt{\beta} = 1$ to illustrate the direct tradeoff between them: as α increases, β must decrease, and vice-versa.

4. If X has an Exponential(λ) distribution so that $f(x|\lambda) = \lambda^{-1}e^{-x/\lambda}$, then the Fisher information (as done in class on January 26) is

$$I(\lambda) = \frac{1}{\lambda^2}$$

and the maximum likelihood estimator (as done in class on January 31) is

$$\hat{\lambda}_{\text{MLE}} = \overline{X}.$$

Hence, a significance level 0.1 test of $H_0 : \lambda = 1/10$ vs. $H_A : \lambda \neq 1/10$ has rejection region

$$\sqrt{nI(\hat{\lambda}_{\text{MLE}})}\,\left|\hat{\lambda}_{\text{MLE}} - \lambda_0\right| \geq z_{0.05}$$

or

$$\frac{\sqrt{n}}{\overline{X}}\left|\overline{X} - \frac{1}{10}\right| \geq 1.645.$$

10.10 Let μ denote the average hardness index. In order to test the manufacturer's claim, we want to test $H_0 : \mu \geq 64$ against $H_A : \mu < 64$. It is equivalent to test $H_0 : \mu = 64$ against $H_A : \mu < 64$. The test statistic is given by

$$Z = \frac{\overline{Y} - \mu_0}{\sigma/\sqrt{n}} = \frac{62 - 64}{8/\sqrt{50}} \approx -1.77.$$

In order to conduct this test at the significance level $\alpha = 0.01$, we find that the rejection region is

$$RR = \{\text{reject } H_0 \text{ if } Z < z_{0.01} = -2.326\}.$$

Since $Z = -1.77$ does not fall in the rejection region $(-1.77 > -2.326)$, we do not reject H_0 in favour of H_A at the 0.01 level. Thus, we conclude that there is insufficient evidence to reject the manufacturer's claim.

10.19 This was discussed in class on Monday, February 28.

10.38 The rejection region is

$$\frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} < -z_\alpha$$

which is true if and only if

$$\hat{\theta} + z_\alpha \sigma_{\hat{\theta}} < \theta_0.$$

That is, H_0 will be rejected at the significance level α if and only if the $100(1-\alpha)\%$ upper confidence bound for θ (namely, $\hat{\theta} + z_\alpha \sigma_{\hat{\theta}}$) is less than θ_0.