Stat 252.01 Winter 2005
Assignment #3 Solutions

(8.9) (a) Let $\theta = \text{Var}(Y)$, and $\hat{\theta} = n(Y/n)(1 - Y/n)$. To prove $\hat{\theta}$ is unbiased, we must show that $\mathbb{E}(\hat{\theta}) \neq \theta$. Since

$$\mathbb{E}(\hat{\theta}) = \mathbb{E}(n(Y/n)(1 - Y/n)) = \mathbb{E}(Y) - \frac{1}{n}\mathbb{E}(Y^2),$$

and since Y is Binomial(n, p) so that $\mathbb{E}(Y) = np$, $\mathbb{E}(Y^2) = \text{Var}(Y) + [\mathbb{E}(Y)]^2 = np(1-p) + n^2p^2$, we conclude that

$$\mathbb{E}(\hat{\theta}) = np - \frac{np(1-p) + n^2p^2}{n} = (n-1)p(1-p).$$

(b) As an unbiased estimator, use

$$\frac{n}{n-1}\,\hat{\theta} = n\left(\frac{Y}{n-1}\right)\left(1 - \frac{Y}{n}\right).$$

(8.10) (a) If $\hat{\theta} = \max(Y_1, \ldots, Y_n)$, then its distribution function is

$$F(t) = \theta^{-n\alpha}\, t^{n\alpha}$$

so that

$$f(t) = n\,\alpha\,\theta^{-n\alpha}\, t^{n\alpha - 1}, \quad 0 \le y \le \theta.$$

We easily calculate that

$$\mathbb{E}(\hat{\theta}) = \int_0^\theta n\,\alpha\,\theta^{-n\alpha}\, t^{n\alpha}\, dt = \frac{n\,\alpha\,\theta^{-n\alpha}\,\theta^{n\alpha+1}}{n\alpha + 1} = \frac{n\,\alpha}{n\,\alpha + 1}\,\theta.$$

Thus, we conclude $\hat{\theta}$ is a biased estimator of θ.

(b) Clearly, the estimator

$$\frac{n\,\alpha + 1}{n\,\alpha}\,\hat{\theta} = \frac{n\,\alpha + 1}{n\,\alpha}\,\max(Y_1, \ldots, Y_n)$$

is an unbiased estimator of θ.

(c) In order to calculate MSE$(\hat{\theta})$ we must find Var$(\hat{\theta})$. We find

$$\mathbb{E}(\hat{\theta}^2) = \int_0^\theta n\,\alpha\,\theta^{-n\alpha}\, t^{n\alpha+1}\, dt = \frac{n\,\alpha\,\theta^{-n\alpha}\,\theta^{n\alpha+2}}{n\alpha + 2} = \frac{n\,\alpha}{n\,\alpha + 2}\,\theta^2.$$

Thus,

$$\text{Var}(\hat{\theta}) = \mathbb{E}(\hat{\theta}^2) - [\mathbb{E}(\hat{\theta})]^2 = \frac{n\,\alpha}{n\,\alpha + 2}\,\theta^2 - \left[\frac{n\,\alpha}{n\,\alpha + 1}\,\theta\right]^2 = \frac{n\alpha}{(n\alpha+1)^2(n\alpha+2)}\,\theta^2.$$

Finally,

$$\text{MSE}(\hat{\theta}) = \text{Var}(\hat{\theta}) + [B(\hat{\theta})]^2 = \left[\frac{n\alpha}{(n\alpha+1)^2(n\alpha+2)} + \frac{1}{(n\alpha+1)^2}\right]\theta^2 = \frac{2\theta^2}{(n\alpha+1)(n\alpha+2)}.$$

(8.34) Let $\theta = V(Y)$. If Y is a geometric random variable, then

$$\mathbb{E}(Y^2) = V(Y) + [\mathbb{E}(Y)]^2 = \frac{2}{p^2} - \frac{1}{p}.$$

Now a little thought shows that

$$\mathbb{E}\left(\frac{Y^2}{2} - \frac{Y}{2}\right) = \frac{1}{p^2} - \frac{1}{2p} - \frac{1}{2p} = \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2} = \theta.$$

Thus, choose

$$\hat{V}(Y) = \hat{\theta} = \frac{Y^2 - Y}{2}.$$

If Y is used to estimate $1/p$, then a two standard error bound on the error of estimation is given by

$$2\sqrt{\hat{V}(Y)} = 2\sqrt{\hat{\theta}} = 2\sqrt{\frac{Y^2 - Y}{2}}.$$

(9.1) Using the results of Exercise 8.4, we find

$$\mathrm{Var}(\hat{\theta}_1) = \theta^2, \quad \mathrm{Var}(\hat{\theta}_2) = \frac{\theta^2}{2}, \quad \mathrm{Var}(\hat{\theta}_3) = \frac{5\theta^2}{9}, \quad \mathrm{Var}(\hat{\theta}_5) = \frac{\theta^2}{3}.$$

Thus,

$$\mathrm{eff}(\hat{\theta}_1, \hat{\theta}_5) = \frac{\mathrm{Var}(\hat{\theta}_5)}{\mathrm{Var}(\hat{\theta}_1)} = \frac{1}{3}, \quad \mathrm{eff}(\hat{\theta}_2, \hat{\theta}_5) = \frac{\mathrm{Var}(\hat{\theta}_5)}{\mathrm{Var}(\hat{\theta}_2)} = \frac{2}{3}, \quad \mathrm{eff}(\hat{\theta}_3, \hat{\theta}_5) = \frac{\mathrm{Var}(\hat{\theta}_5)}{\mathrm{Var}(\hat{\theta}_3)} = \frac{3}{5}.$$

(9.4) In Example 9.1, it is shown that

$$\mathrm{Var}(\hat{\theta}_2) = \frac{\theta^2}{n(n+2)},$$

and we have as a simple extension of Problem #1 on Assignment #2 that

$$\mathrm{Var}(\hat{\theta}_1) = (n+1)^2\,\mathrm{Var}(Y_{(1)}) = (n+1)^2\left[\frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2}\right] = \frac{n\theta^2}{n+2}.$$

Thus we conclude,

$$\mathrm{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\mathrm{Var}(\hat{\theta}_2)}{\mathrm{Var}(\hat{\theta}_1)} = \frac{1}{n^2}.$$

Notice that this result implies that

$$\mathrm{Var}(\hat{\theta}_1) = n^2\,\mathrm{Var}(\hat{\theta}_2).$$

As n increases, the variance of $\hat{\theta}_1$ increases very quickly relative to the variance of $\hat{\theta}_2$. In other words, the larger n, the bigger the variance of $\hat{\theta}_1$ relative to variance $\hat{\theta}_2$. Thus, $\hat{\theta}_2$ is a markedly superior (unbiased) estimator.

(9.7) If $\mathrm{MSE}(\hat{\theta}_1) = \theta^2$, then $\mathrm{Var}(\hat{\theta}_1) = \mathrm{MSE}(\hat{\theta}_1) = \theta^2$ since $\hat{\theta}_1$ is unbiased. If $\hat{\theta}_2 = \overline{Y}$, then since the Y_i are exponential, we conclude $\mathbb{E}(\overline{Y}) = \theta$ and $\mathrm{Var}(\overline{Y}) = \theta^2/n$. Thus,

$$\mathrm{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\mathrm{Var}(\hat{\theta}_2)}{\mathrm{Var}(\hat{\theta}_1)} = \frac{1}{n}.$$

(9.62) If Y has a Poisson distribution, then $\mathbb{E}(Y) = \lambda$. Hence, the method of moments implies that $\mu_1 = \hat{\mu}_1$ or

$$\hat{\lambda}_{\text{MOM}} = \overline{Y}.$$

(9.64) If Y has a normal distribution, then $\mathbb{E}(Y) = \mu$ and $\mathbb{E}(Y^2) = \mu^2 + \sigma^2$. (Think back to Assignment #1.) Hence, the method of moments implies that $\mu_1 = \hat{\mu}_1$ and $\mu_2 = \hat{\mu}_2$ so that

$$\mu = \overline{Y} \quad \text{and} \quad \mu^2 + \sigma^2 = \frac{1}{n}\sum Y_i^2.$$

Solving this system of equation gives

$$\hat{\mu}_{\text{MOM}} = \overline{Y} \quad \text{and} \quad \hat{\sigma}^2_{\text{MOM}} = \frac{1}{n}\sum Y_i^2 - \overline{Y}^2 = \frac{1}{n}\sum (Y_i - \overline{Y})^2 = \frac{n-1}{n}\, S^2.$$

(9.66) (a) If Y has density $f(y|\theta) = 2\theta^{-2}(\theta - y)$ for $0 \le y \le \theta$, then

$$\mathbb{E}(Y) = \int_0^\theta 2\theta^{-2} y(\theta - y)\, dy = \frac{\theta}{3}.$$

Thus, the method of moments implies that $\mu_1 = \hat{\mu}_1$ or

$$\hat{\theta}_{\text{MOM}} = 3\overline{Y}.$$

2. (a) It is highly unlikely that the iid assumption is reasonable. In order to postulate iid $\text{Bin}(k, p)$, she is assuming that each animal has the same probability of being trapped. This is doubtful both within a species and between species. (Are some animals "dumber" and others "smarter"? What about different species? Are some more cautious than others?) This is also doubtful because animals are likely to get "smarter" after being trapped once. (Think of any Pavlovian experiment.) The independent trials assumption is also dubious. Is it reasonable to assume that animals do not warn others of the danger of the trap? Probably not.

(b) For a $\text{Bin}(k, p)$ random variable Y, we have $\mathbb{E}(Y) = kp$ and $\mathbb{E}(Y^2) = \text{Var}(Y) + [\mathbb{E}(Y)]^2 = kp(1-p) + k^2p^2$. The method of moments system implies that $\hat{\mu}_1 = kp$ and $\hat{\mu}_2 = kp(1-p) + k^2p^2$. Solving gives

$$\hat{p}_{\text{MOM}} = 1 - \frac{\hat{\mu}_2 - (\hat{\mu}_1)^2}{\hat{\mu}_1}$$

and

$$\hat{k}_{\text{MOM}} = \frac{\hat{\mu}_1}{\hat{p}_{\text{MOM}}}.$$

The data yield $\hat{\mu}_1 = 12.6$ and $\hat{\mu}_2 = 163$. Thus,

$$\hat{p}_{\text{MOM}} = \frac{209}{315} \approx 0.663 \quad \text{and} \quad \hat{k}_{\text{MOM}} = \frac{3969}{209} \approx 19.$$

(c) In this case the data yield $\hat{\mu}_1 = 11.2$ and $\hat{\mu}_2 = 139.2$ which give

$$\hat{p}_{\text{MOM}} = \frac{-8}{35} \approx -0.229 \quad \text{and} \quad \hat{k}_{\text{MOM}} = -49.$$

These are nonsensical estimates since we require $p \in [0, 1]$ and $k > 0$. Clearly if these were the data observed, the postulate of a binomial distribution would definitely be cast into serious doubt!