Mathematics/Statistics 251 Fall 2015 Midterm #1 – Solutions

1. If X denotes the number of prepackaged lunch snacks that contain harmful bacteria, then

(a)

$$\mathbf{P}(X=0) = \binom{5}{0}\left(\frac{3}{1000}\right)^0\left(\frac{997}{1000}\right)^5 = \left(\frac{997}{1000}\right)^5,$$

and

(b)

$$\begin{aligned}\mathbf{P}(X\geq 3) &= \mathbf{P}(X=3)+\mathbf{P}(X=4)+\mathbf{P}(X=5)\\ &= \binom{5}{3}\left(\frac{3}{1000}\right)^3\left(\frac{997}{1000}\right)^2+\binom{5}{4}\left(\frac{3}{1000}\right)^4\left(\frac{997}{1000}\right)^1+\binom{5}{5}\left(\frac{3}{1000}\right)^5\left(\frac{997}{1000}\right)^0.\end{aligned}$$

2. (a) Since A and B are disjoint, we have $\mathbf{P}(A\cap B)=0$. Therefore,

$$\begin{aligned}\mathbf{P}(A\cup B) &= \mathbf{P}(A)+\mathbf{P}(B)-\mathbf{P}(A\cap B)\\ &= \mathbf{P}(A)+\mathbf{P}(B) = 1-\mathbf{P}(A^c)+1-\mathbf{P}(B^c)\\ &= (1-0.4)+(1-0.9)\\ &= 0.6+0.1\\ &= 0.7.\end{aligned}$$

2. (b) Since A and B are independent, we have $\mathbf{P}(A\cap B)=\mathbf{P}(A)\mathbf{P}(B)$. Therefore,

$$\begin{aligned}\mathbf{P}(A\cup B) = \mathbf{P}(A)+\mathbf{P}(B)-\mathbf{P}(A\cap B) &= \mathbf{P}(A)+\frac{\mathbf{P}(A\cap B)}{\mathbf{P}(B)}-\mathbf{P}(A\cap B)\\ &= 0.6+\frac{0.3}{0.6}-0.3\\ &= 0.6+0.5-0.3\\ &= 0.8.\end{aligned}$$

3. Since A and B are independent, we have $\mathbf{P}(A\cap B)=\mathbf{P}(A)\mathbf{P}(B)$. To show that A^c and B^c are also independent, we must show that $\mathbf{P}(A^c\cap B^c)=\mathbf{P}(A^c)\mathbf{P}(B^c)$. Therefore, since

$$\begin{aligned}\mathbf{P}(A^c\cap B^c) = \mathbf{P}((A\cup B)^c) = 1-\mathbf{P}(A\cup B) &= 1-[\mathbf{P}(A)+\mathbf{P}(B)-\mathbf{P}(A\cap B)]\\ &= 1-\mathbf{P}(A)-\mathbf{P}(B)+\mathbf{P}(A\cap B)\\ &= 1-\mathbf{P}(A)-\mathbf{P}(B)+\mathbf{P}(A)\mathbf{P}(B)\\ &= [1-\mathbf{P}(A)][1-\mathbf{P}(B)]\\ &= \mathbf{P}(A^c)\mathbf{P}(B^c)\end{aligned}$$

we conclude A^c and B^c are independent.

4. Let W_j, B_j, and R_j denote the events that the jth sock Andrew selects will be white, blue, or red, respectively. Therefore,

(a)

$$\mathbf{P}\left(\text{both socks are blue}\right) = \mathbf{P}\left(B_1 \cap B_2\right) = \mathbf{P}\left(B_1\right)\mathbf{P}\left(B_2 | B_1\right) = \frac{4}{14} \cdot \frac{3}{13} = \frac{6}{91},$$

(b)

$$\begin{aligned}
&\mathbf{P}\left(\text{both socks are same colour}\right)\\
&\qquad = \mathbf{P}\left(W_1 \cap W_2 \text{ or } B_1 \cap B_2 \text{ or } R_1 \cap R_2\right)\\
&\qquad = \mathbf{P}\left(W_1 \cap W_2\right) + \mathbf{P}\left(B_1 \cap B_2\right) + \mathbf{P}\left(R_1 \cap R_2\right)\\
&\qquad = \mathbf{P}\left(W_1\right)\mathbf{P}\left(W_2|W_1\right) + \mathbf{P}\left(B_1\right)\mathbf{P}\left(B_2|B_1\right) + \mathbf{P}\left(R_1\right)\mathbf{P}\left(R_2|R_1\right)\\
&\qquad = \frac{2}{14} \cdot \frac{1}{13} + \frac{4}{14} \cdot \frac{3}{13} + \frac{8}{14} \cdot \frac{7}{13}\\
&\qquad = \frac{35}{91} = \frac{5}{13},
\end{aligned}$$

and

(c)

$$\begin{aligned}
&\mathbf{P}\left(\text{both socks are blue} \mid \text{both socks are same colour}\right)\\
&= \frac{\mathbf{P}\left(\text{both socks are same colour} \mid \text{both socks are blue}\right)\mathbf{P}\left(\text{both socks are blue}\right)}{\mathbf{P}\left(\text{both socks are same colour}\right)}\\
&= \frac{1 \cdot \frac{6}{91}}{\frac{35}{91}}\\
&= \frac{6}{35}.
\end{aligned}$$

5. (a) Since F is a distribution function, it is necessarily the case that $F(2) = 1$. This implies that $4c = 1$ so $c = 1/4$.

5. (b) Since $f(x) = F'(x)$, we obtain

$$f(x) = \begin{cases} 2x, & 0 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

5. (c) We have $\mathbf{P}\left(1 < X \le 2\right) = \mathbf{P}\left(X \le 2\right) - \mathbf{P}\left(X \le 1\right) = F(2) - F(1) = 1 - \frac{1}{4} = \frac{3}{4}$.