Math/Stat 251 Fall 2015
Binomial Probabilities (September 30, 2015 and October 2, 2015)

Suppose that we are considering repeated trials where in each trial we might observe one of only two possible results. We can label these results as *success* or *failure*. (Or, in applications, they might be on/off, zero/one, yes/no, left/right, male/female, shows improvement/does not show improvement, etc.) Assume further that the result of each trial is independent of the results of any other trial and that the probability of success in any given trial is p (so that the probability of failure on any given trial is $1-p$).

If we repeat the trials a total of n times and count the *total number of successes* observed, then the probability that exactly k successes occur is computed as follows. In order for there to be exactly k successes, then there must be $n-k$ failures. Any particular sequence of k successes and $n-k$ failures occurs with probability $p^k(1-p)^{n-k}$. Since we are interested in just the total number of successes k and not a particular ordering, we need to count the number of ways to arrange k successes among n trials. There are $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ ways to do this. Thus,

$$\mathbf{P}\left(\text{exactly } k \text{ successes in } n \text{ trials}\right) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

for $k = 0, 1, 2, \ldots, n$.

Example. What is the probability that we observe exactly $k=3$ successes in $n=5$ trials? We are going to answer this question by writing out in detail in order to explain the formula above. Let S_j be the event that a success is observed on the jth trial so that $\mathbf{P}(S_j) = p$. Let $F_j = S_j^c$ be the event that a failure is observed on the jth trial so that $\mathbf{P}(F_j) = 1-p$. Hence,

$$\begin{aligned}
&\mathbf{P}\left(\text{exactly 3 successes in 5 trials}\right)\\
&\quad= \mathbf{P}\{S_1S_2S_3F_4F_5 \text{ or } S_1S_2F_3S_4F_5 \text{ or } S_1S_2F_3F_4S_5 \text{ or } S_1F_2S_3S_4F_5 \text{ or } S_1F_2S_3F_4S_5 \text{ or}\\
&\qquad\qquad S_1F_2F_3S_4S_5 \text{ or } F_1S_2S_3S_4F_5 \text{ or } F_1S_2S_3F_4S_5 \text{ or } F_1S_2F_3S_4S_5 \text{ or } F_1F_2S_3S_4S_5\}\\
&\quad= \mathbf{P}(S_1S_2S_3F_4F_5) + \mathbf{P}(S_1S_2F_3S_4F_5) + \mathbf{P}(S_1S_2F_3F_4S_5) + \mathbf{P}(S_1F_2S_3S_4F_5)\\
&\qquad\qquad + \mathbf{P}(S_1F_2S_3F_4S_5) + \mathbf{P}(S_1F_2F_3S_4S_5) + \mathbf{P}(F_1S_2S_3S_4F_5)\\
&\qquad\qquad\qquad + \mathbf{P}(F_1S_2S_3F_4S_5) + \mathbf{P}(F_1S_2F_3S_4S_5) + \mathbf{P}(F_1F_2S_3S_4S_5)\\
&\quad= \mathbf{P}(S_1)\mathbf{P}(S_2)\mathbf{P}(S_3)\mathbf{P}(F_4)\mathbf{P}(F_5) + \mathbf{P}(S_1)\mathbf{P}(S_2)\mathbf{P}(F_3)\mathbf{P}(S_4)\mathbf{P}(F_5)\\
&\qquad + \mathbf{P}(S_1)\mathbf{P}(S_2)\mathbf{P}(F_3)\mathbf{P}(F_4)\mathbf{P}(S_5) + \mathbf{P}(S_1)\mathbf{P}(F_2)\mathbf{P}(S_3)\mathbf{P}(S_4)\mathbf{P}(F_5)\\
&\qquad\quad + \mathbf{P}(S_1)\mathbf{P}(F_2)\mathbf{P}(S_3)\mathbf{P}(F_4)\mathbf{P}(S_5) + \mathbf{P}(S_1)\mathbf{P}(F_2)\mathbf{P}(F_3)\mathbf{P}(S_4)\mathbf{P}(S_5)\\
&\qquad\qquad + \mathbf{P}(F_1)\mathbf{P}(S_2)\mathbf{P}(S_3)\mathbf{P}(S_4)\mathbf{P}(F_5) + \mathbf{P}(F_1)\mathbf{P}(S_2)\mathbf{P}(S_3)\mathbf{P}(F_4)\mathbf{P}(S_5)\\
&\qquad\qquad\quad + \mathbf{P}(F_1)\mathbf{P}(S_2)\mathbf{P}(F_3)\mathbf{P}(S_4)\mathbf{P}(S_5) + \mathbf{P}(F_1)\mathbf{P}(F_2)\mathbf{P}(S_3)\mathbf{P}(S_4)\mathbf{P}(S_5)\\
&\quad= p^3(1-p)^2 + p^3(1-p)^2 + \cdots + p^3(1-p)^2\\
&\quad= 10p^3(1-p)^2\\
&\quad= \binom{5}{3} p^3(1-p)^2.
\end{aligned}$$

Example. Suppose that a fair coin was tossed 20 times. What is the probability that exactly 12 heads were observed?

Solution. If we define *success* as *observe a head*, then the binomial probability formula tells us that

$$\mathbf{P}\left(\text{exactly12 heads in 20 tosses}\right) = \binom{20}{12}(1/2)^{12}(1/2)^{8} = 2^{-20}\binom{20}{12}.$$

Example. Suppose that a fair coin was tossed 20 times and that there were 12 heads observed. (You may assume that the results of subsequent tosses were independent.)

(a) What is the probability that the first toss showed heads?

(b) What is the probability that the first two tosses showed heads?

(c) What is the probability that at least two of the first five tosses landed heads?

Solution. Let X denote the number of heads observed in 20 tosses of the coin.

(a) Using Bayes' rule, we find

$$\begin{aligned}&\mathbf{P}\left(\text{1st toss showed heads} \,|\, X = 12\right)\\ &\qquad = \frac{\mathbf{P}\left(X = 12 \,|\, \text{1st toss showed heads}\right)\mathbf{P}\left(\text{1st toss showed heads}\right)}{\mathbf{P}\left(X = 12\right)}.\end{aligned}$$

Clearly, from the solution to the previous example,

$$\mathbf{P}\left(X = 12\right) = \binom{20}{12}(1/2)^{12}(1/2)^{8}$$

and

$$\mathbf{P}\left(\text{1st toss showed heads}\right) = \frac{1}{2}.$$

Notice that if the 1st toss showed heads, then the only way for $X = 12$ is if there are 11 heads in the remaining 19 tosses. Thus,

$$\mathbf{P}\left(X = 12 \,|\, \text{1st toss showed heads}\right) = \binom{19}{11}(1/2)^{11}(1/2)^{9}.$$

Combining everything gives

$$\mathbf{P}\left(\text{1st toss showed heads} \,|\, X = 12\right) = \frac{\binom{19}{11}(1/2)^{11}(1/2)^{9}\cdot(1/2)}{\binom{20}{12}(1/2)^{12}(1/2)^{8}} = \frac{\binom{19}{11}}{\binom{20}{12}} = \frac{12}{20}.$$

(b) Using Bayes' rule, we find as in **(a)** that

$$\begin{aligned}&\mathbf{P}\left(\text{1st two tosses showed heads} \,|\, X = 12\right)\\ &\qquad = \frac{\mathbf{P}\left(X = 12 \,|\, \text{1st two tosses showed heads}\right)\mathbf{P}\left(\text{1st two tosses showed heads}\right)}{\mathbf{P}\left(X = 12\right)}.\end{aligned}$$

If the 1st two tosses showed heads, then the only way for $X = 12$ is if there are 10 heads in the remaining 18 tosses. Thus,

$$\mathbf{P}\left(X = 12 \,|\, \text{1st two tosses showed heads}\right) = \binom{18}{10}(1/2)^{10}(1/2)^{8}.$$

Thus,

$$\mathbf{P}\left(\text{1st two tosses showed heads} \,|\, X = 12\right) = \frac{\binom{18}{10}(1/2)^{10}(1/2)^{8} \cdot (1/2)^{2}}{\binom{20}{12}(1/2)^{12}(1/2)^{8}} = \frac{\binom{18}{10}}{\binom{20}{12}} = \frac{12}{20} \cdot \frac{11}{19}.$$

(c) Observe that the event {at least 2 of the first 5 tosses landed heads} can be written as the union of the events

$$\{\text{exactly 2 of the first 5 landed heads}\} \cup \cdots \cup \{\text{exactly 5 of the first 5 landed heads}\}$$

Thus, using Bayes' rule, we find

$$\begin{aligned}&\mathbf{P}\left(\text{exactly 2 of the first 5 landed heads} \,|\, X = 12\right)\\ &\quad = \frac{\mathbf{P}\left(X = 12 \,|\, \text{exactly 2 of the first 5 landed heads}\right)\mathbf{P}\left(\text{exactly 2 of the first 5 landed heads}\right)}{\mathbf{P}\left(X = 12\right)}.\end{aligned}$$

Notice that

$$\mathbf{P}\left(\text{exactly 2 of the first 5 landed heads}\right) = \binom{5}{2}(1/2)^{2}(1/2)^{3}.$$

Furthermore, if exactly 2 of the first 5 tosses showed heads, then the only way for $X = 12$ is if there are 10 heads in the remaining 15 tosses; that is,

$$\mathbf{P}\left(X = 12 \,|\, \text{exactly 2 of the first 5 landed heads}\right) = \binom{15}{10}(1/2)^{10}(1/2)^{5}.$$

Combined, we conclude

$$\mathbf{P}\left(\text{exactly 2 of the first 5 landed heads} \,|\, X = 12\right) = \frac{\binom{15}{10}(1/2)^{10}(1/2)^{5} \cdot \binom{5}{2}(1/2)^{2}(1/2)^{3}}{\binom{20}{12}(1/2)^{12}(1/2)^{8}} = \frac{\binom{15}{10} \cdot \binom{5}{2}}{\binom{20}{12}}.$$

Similarly, we can find the probability that exactly k of the first 5 tosses landed heads given that $X = 12$. Piecing everything back together gives

$$\mathbf{P}\left(\text{at least 2 of the first 5 landed heads} \,|\, X = 12\right) = \frac{\binom{15}{10} \cdot \binom{5}{2}}{\binom{20}{12}} + \frac{\binom{15}{9} \cdot \binom{5}{3}}{\binom{20}{12}} + \frac{\binom{15}{8} \cdot \binom{5}{4}}{\binom{20}{12}} + \frac{\binom{15}{7} \cdot \binom{5}{5}}{\binom{20}{12}}.$$