Solutions

Problem 1. Observe that

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x.$$

Now the substitution $u = x^2/2$, $\mathrm{d}u = x\,\mathrm{d}x$ implies that

$$\int x e^{-x^2/2}\,\mathrm{d}x = \int e^{-u}\,\mathrm{d}u = -e^{-u} = -e^{-x^2/2}$$

and so

$$\begin{aligned}\int_{-\infty}^{\infty} x e^{-x^2/2}\,\mathrm{d}x &= \lim_{a\to-\infty}\int_a^0 x e^{-x^2/2}\,\mathrm{d}x + \lim_{b\to\infty}\int_0^b x e^{-x^2/2}\,\mathrm{d}x \\ &= \lim_{a\to-\infty}[1 - e^{-a^2/2}] + \lim_{b\to\infty}[e^{-b^2/2} - 1] \\ &= 0.\end{aligned}$$

Thus, $\mathbb{E}(X) = 0$

Problem 2. Observe that

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x.$$

To evaluate

$$\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,\mathrm{d}x$$

use integration-by-parts with $u = x$ and $\mathrm{d}v = x e^{-x^2/2}\,\mathrm{d}x$ so that

$$\int x^2 e^{-x^2/2}\,\mathrm{d}x = -x e^{-x^2} + e^{-x^2/2}\,\mathrm{d}x$$

Thus,

$$\begin{aligned}\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,\mathrm{d}x &= \lim_{a\to-\infty}\int_a^0 x^2 e^{-x^2/2}\,\mathrm{d}x + \lim_{b\to\infty}\int_0^b x^2 e^{-x^2/2}\,\mathrm{d}x \\ &= \lim_{a\to-\infty}[-a e^{-a^2}] + \lim_{b\to\infty}[b e^{-b^2}] + \int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x.\end{aligned}$$

If we multiply by $1/\sqrt{2\pi}$, then using the fact that the density function of a $\mathcal{N}(0,1)$ random variable integrates to 1, we conclude

$$\mathbb{E}(X^2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x = 1.$$

Problem 3. Using the results of the previous two problems, we find

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = 1 - 0^2 = 1.$$

Problem 4. By definition,

$$m_X(t) = \mathbb{E}(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - x^2/2}\,\mathrm{d}x.$$

Now, we complete the square in the exponent; that is, we write

$$tx - \frac{x^2}{2} = -\frac{1}{2}(x^2 - 2tx) = -\frac{1}{2}(x^2 - 2tx + t^2 - t^2) = -\frac{1}{2}(x^2 - 2tx + t^2) + \frac{t^2}{2} = -\frac{(x-t)^2}{2} + \frac{t^2}{2}$$

so that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - x^2/2}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2} + \frac{t^2}{2}}\,\mathrm{d}x = e^{t^2/2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2}\,\mathrm{d}x.$$

However, if we substitute $u = x - t$, then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2}\,\mathrm{d}u = 1$$

(since it is the integral of the density function of a $\mathcal{N}(0,1)$ random variable), and so we conclude

$$m_X(t) = e^{t^2/2}.$$

Problem 5. If $Y = \sigma X + \mu$, then

$$\mathbb{E}(Y) = \mathbb{E}(\sigma X + \mu) = \sigma \mathbb{E}(X) + \mu = \mu$$

and

$$\mathbb{E}(Y^2) = \mathbb{E}[(\sigma X + \mu)^2] = \mathbb{E}(\sigma^2 X^2 + 2\sigma\mu X + \mu^2) = \sigma^2 \mathbb{E}(X^2) + 2\sigma\mu \mathbb{E}(X) + \mu^2 = \sigma^2 + \mu^2$$

since $\mathbb{E}(X) = 0$ and $\mathbb{E}(X) = 1$ so that

$$\mathrm{Var}(Y) = \mathbb{E}(Y^2) - [\mathbb{E}(Y)]^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2.$$

Problem 6. If $Y = \sigma X + \mu$, then for any $y \in \mathbb{R}$, the distribution function of Y is

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(\sigma X + \mu \le y) = \mathbf{P}\left(X \le \frac{y-\mu}{\sigma}\right) = \int_{-\infty}^{\frac{y-\mu}{\sigma}} f_X(x)\,\mathrm{d}x$$

so that

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right) \cdot \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}.$$

Problem 7. If $Y = \sigma X + \mu$, then

$$m_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t(\sigma X+\mu)}] = \mathbb{E}[e^{t\mu}e^{\sigma t X}] = e^{t\mu}\mathbb{E}[e^{\sigma t X}] = e^{t\mu}m_X(\sigma t) = e^{t\mu}e^{(\sigma t)^2/2} = e^{t\mu+\sigma^2t^2/2}.$$

Problem 8. Since $Y \sim \mathcal{N}(\mu, \sigma^2)$ and

$$Z = \frac{Y-\mu}{\sigma} = \frac{1}{\sigma}Y - \frac{\mu}{\sigma},$$

then the moment generating function of Z is

$$m_Z(t) = \mathbb{E}[e^{tZ}] = \mathbb{E}\left[e^{t\left(\frac{1}{\sigma}Y-\frac{\mu}{\sigma}\right)}\right] = e^{-t\mu/\sigma}m_Y(t/\sigma) = e^{-t\mu/\sigma}e^{t\mu/\sigma+\sigma^2t^2/(2\sigma^2)} = e^{t^2/2}$$

which is the moment generating function of a $\mathcal{N}(0,1)$ random variable. Thus, $Z \sim \mathcal{N}(0,1)$.

Problem 9. If $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, then from Problem 7 we know

$$m_{X_i}(t) = \mathbb{E}(e^{tX_i}) = e^{t\mu_i+\sigma_i^2t^2/2}.$$

Therefore, the mgf of $X_1 + X_2$ is

$$\begin{aligned} m_{X_1+X_2}(t) = \mathbb{E}[e^{t(X_1+X_2)}] = \mathbb{E}[e^{tX_1}e^{tX_2}] = \mathbb{E}(e^{tX_1})\mathbb{E}(e^{tX_2}) &= e^{t\mu_1+\sigma_1^2t^2/2}e^{t\mu_1+\sigma_1^2t^2/2} \\ &= e^{t(\mu_1+\mu_2)+(\sigma_1^2+\sigma_2^2)t^2/2} \end{aligned}$$

which we recognize as the mgf of a $\mathcal{N}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$ random variable.

Problem 10. The mgf of $X_1 + \cdots + X_n$ is

$$\begin{aligned} m_{X_1+\cdots+X_n}(t) = \mathbb{E}[e^{t(X_1+\cdots+X_n)}] = \mathbb{E}[e^{tX_1}\cdots e^{tX_n}] &= \mathbb{E}(e^{tX_1})\cdots\mathbb{E}(e^{tX_n}) \\ &= e^{t\mu_1+\sigma_1^2t^2/2}\cdots e^{t\mu_n+\sigma_n^2t^2/2} \\ &= e^{t(\mu_1+\cdots+\mu_n)+(\sigma_1^2+\cdots+\sigma_n^2)t^2/2} \end{aligned}$$

which we recognize as the mgf of a $\mathcal{N}(\mu_1+\cdots+\mu_n, \sigma_1^2+\cdots+\sigma_n^2)$ random variable.

Problem 11. We know from the previous problem that if $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, then the mgf of $X_1 + \cdots + X_n$ is

$$m_{X_1+\cdots+X_n}(t) = e^{n\mu t+n\sigma^2t^2/2}.$$

Therefore, the mgf of $\overline{X}$ is

$$m_{\overline{X}}(t) = \mathbb{E}(e^{t\overline{X}}) = \mathbb{E}[e^{\frac{t}{n}(X_1+\cdots+X_n)}] = m_{X_1+\cdots+X_n}(t/n) = e^{n\mu(t/n)+n\sigma^2(t/n)^2/2} = e^{\mu t+\sigma^2t^2/(2n)}$$

which we recognize as the mgf of a $\mathcal{N}(\mu, \sigma^2/n)$ random variable. We can now conclude from Problem 8 that

$$\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1).$$